Model Reduction of Complex Dynamical Networks based on State Aggregation

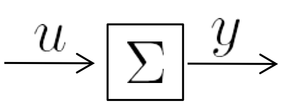
Takayuki Ishizaki (Tokyo Institute of Technology)

Contents

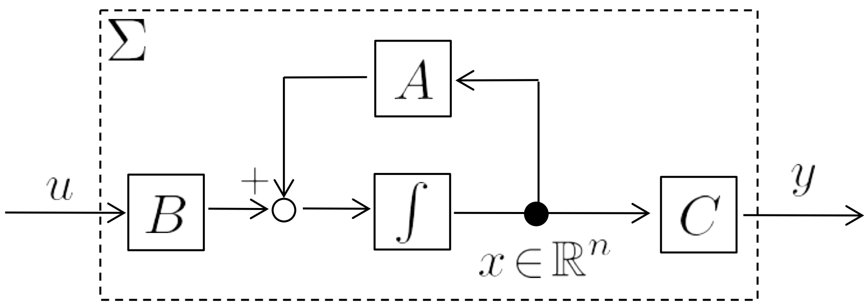
- Overview of model reduction problem (preliminary)
  - what is model reduction?
  - error evaluation in terms of $\mathcal{H}_\infty/\mathcal{H}_2$-norms
- Network structure preserving model reduction (main)
  - system description of dynamical networks
  - key idea of state aggregation with an $\mathcal{H}_\infty$-error evaluation
  - numerical example
- State aggregation with an $\mathcal{H}_2$-error evaluation (additional)
  - controllability gramian

Overview of Model Reduction

Given stable system



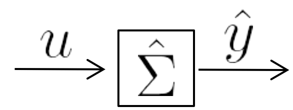
$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$



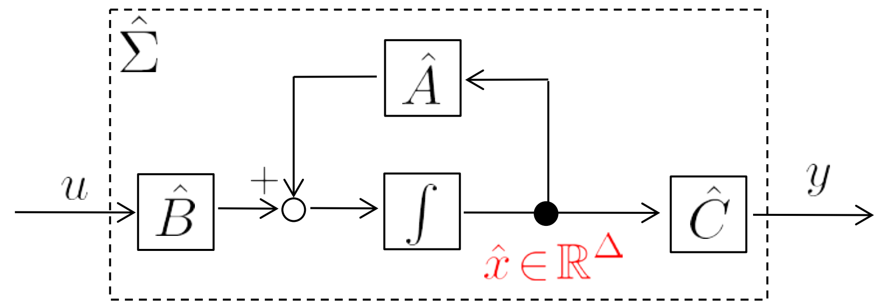
input-to-state map (A, B)

state-to-output map (A, C)

Find Stable reduced model



$$\hat{\Sigma} : \begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \\ \hat{y} = \hat{C}\hat{x} \end{cases}$$



Dimension of state: $\Delta < n$

$$Px = \hat{x}, \ P \in \mathbb{R}^{\Delta \times n} \longrightarrow \hat{A} = PAP^{\dagger}, \ \hat{B} = PB, \ \hat{C} = CP^{\dagger}$$

※ $PP^{\dagger} = I_{\Delta}$

Find P such that $\|\Sigma - \hat{\Sigma}\|$ is small enough

Typical System Norms

※ $\mathscr{L}[*]$: Laplace transform of $*$

$\| * \|_F$: Frobenius norm of $*$

stable system $\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$

solution $y(t) = \int_0^\infty h(t-\tau)u(\tau)d\tau$

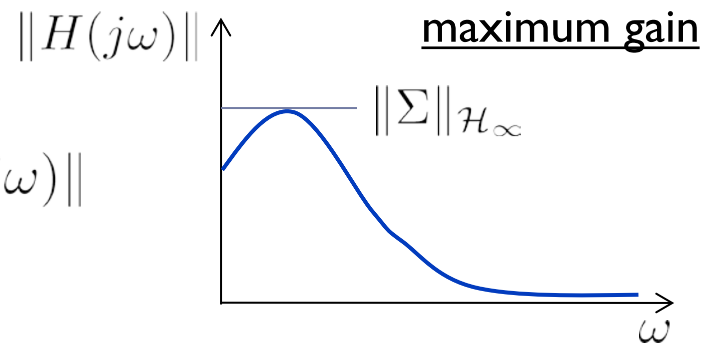
impulse response $h(t) := Ce^{At}B$

transfer function $H(s) := \mathscr{L}[h] = C(sI_n - A)^{-1}B$

<u>$\mathcal{H}_\infty$-norm</u>

$$\|\Sigma\|_{\mathcal{H}_\infty} = \sup_{u \neq 0} \frac{\|y(t)\|_{\mathcal{L}_2}}{\|u(t)\|_{\mathcal{L}_2}} = \sup_{\omega \in \mathbb{R}} \|H(j\omega)\|$$

<u>maximum gain</u>



<u>$\mathcal{H}_2$-norm</u> $\|\Sigma\|_{\mathcal{H}_2} = \|h\|_{\mathcal{L}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|H(j\omega)\|_F^2 \, d\omega \right)^{\frac{1}{2}}$ energy of impulse response

$\mathcal{L}_2$-norm of $f : \mathbb{R}_+ \to \mathbb{R}^{p \times m}$ $\|f\|_{\mathcal{L}_2} := \left(\int_0^\infty \|f(t)\|_F^2 \, dt \right)^{\frac{1}{2}}$

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System Description

[Definition] (single-input) **Dynamical Network** (A, b)

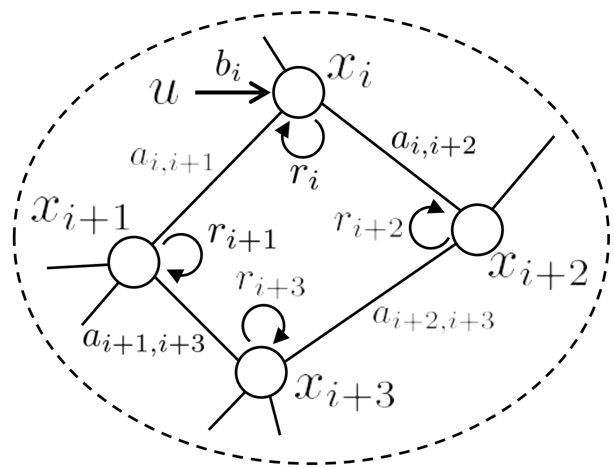
A linear system $\dot{x} = Ax + bu$ with $A = \{a_{i,j}\} \in \mathbb{R}^{n\times n}$ and $b = \{b_i\} \in \mathbb{R}^n$ is said to be a *dynamical network* (A, b) if A is stable and symmetric.

A generalization of reaction-diffusion systems:

$$\dot{x}_i = -r_i x_i + \sum_{j=1, j\neq i}^{n} a_{i,j}(x_j - x_i) + b_i u$$

where $\begin{cases} r_i \geq 0 : \text{intensity of **reaction** (chemical dissolution)} \\ a_{i,j} = a_{j,i} \geq 0 : \text{intensity of **diffusion** between } x_i \text{ and } x_j \end{cases}$

Connected undirected graph



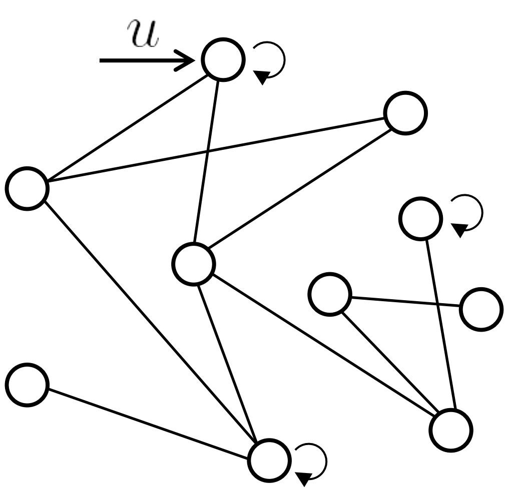
stable, symmetric

Drawback of Traditional Model Reduction Methods

- Traditional model reduction methods
  - Balanced truncation, Krylov projection, Hankel norm approximation
  - No specific structure in transformation matrix P

Drawback: Network structure (spatial information) is destroyed

Given (A, b)



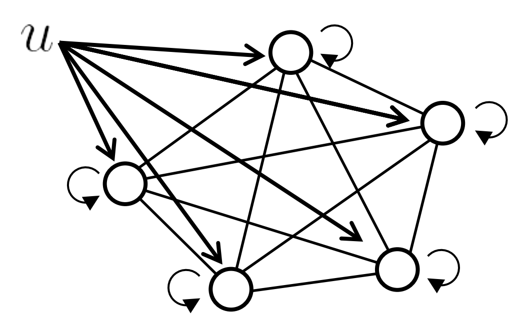
A, b : sparse ☺

$$Px = \hat{x}$$

$\longrightarrow$

$P \in \mathbb{R}^{\Delta \times n}$: full matrix

Reduced model $(PAP^{\dagger}, Pb)$

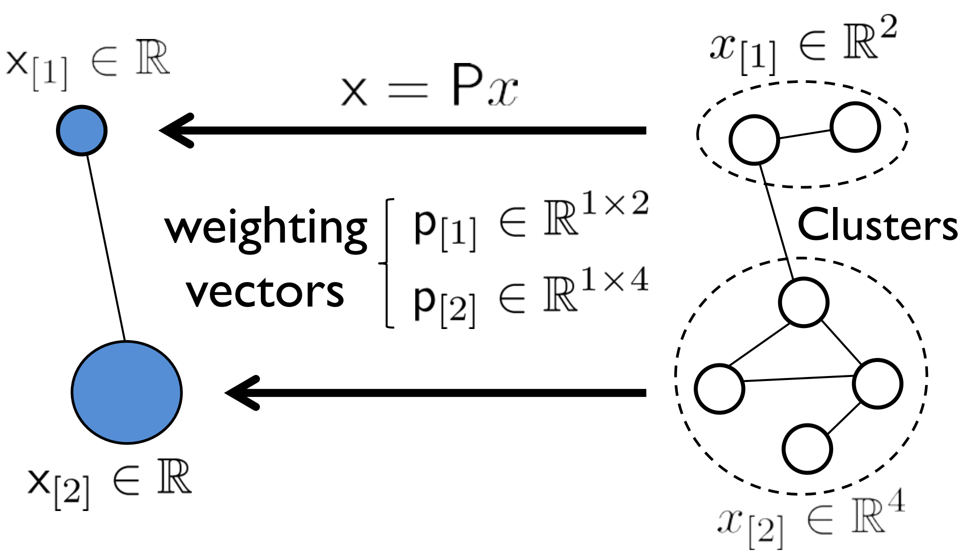


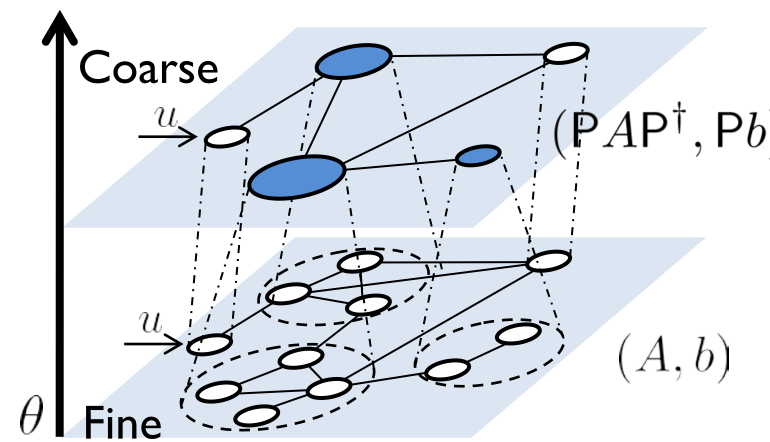
$PAP^{\dagger}, Pb$: dense ☹

no physical interpretation...

State Aggregation Approach

- Model reduction by aggregating disjoint sets of states $\{x_{[l]}\}_{l\in\{1,\dots,L\}}$
  - Block-diagonally structured aggregation matrix $\mathsf{P} = \mathrm{Diag}\left(\mathsf{p}_{[1]},\dots,\mathsf{p}_{[L]}\right)$
  - Network topology among clusters is to be preserved ☺





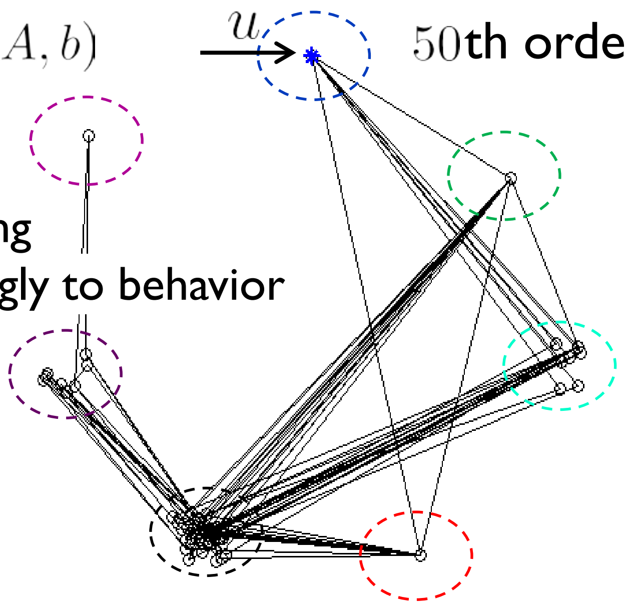
$$\mathsf{x} = \mathsf{P}x \iff \begin{bmatrix} \mathsf{x}_{[1]} \\ \mathsf{x}_{[2]} \end{bmatrix} = \begin{bmatrix} \mathsf{p}_{[1]} & \\ & \mathsf{p}_{[2]} \end{bmatrix} \begin{bmatrix} x_{[1]} \\ x_{[2]} \end{bmatrix}$$

How to find "reducible" clusters?

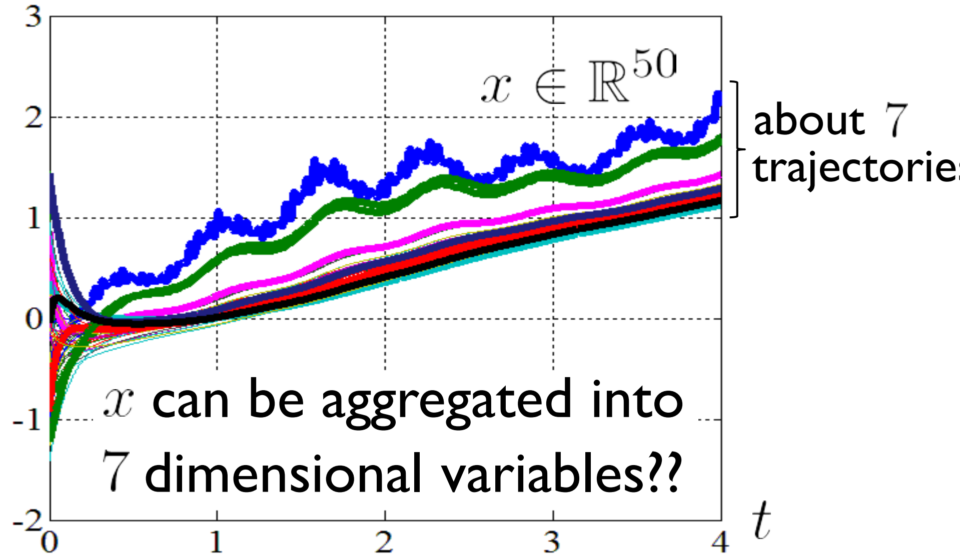
※ For simplicity of notation, suppose the indices are permuted accordingly to clusters

Key Observation to Construct Reducible Clusters

Given (A, b)



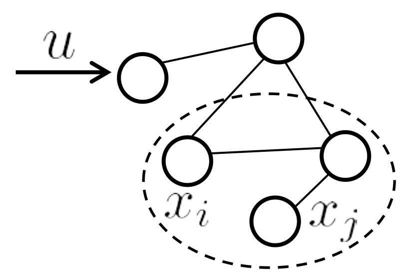
[State behavior under a input signal]



[Definition] **Reducible Cluster**

Let $\mathcal{I}_{[l]}$ be the index set of the l-th cluster.

The l-th cluster is said to be *reducible* if



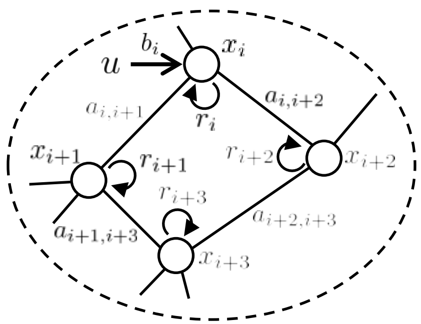
$$\exists \rho_{i,j} \in \mathbb{R} \ \text{ s.t. } \ y_i(t) \equiv \rho_{i,j} y_j(t), \ \forall i, j \in \mathcal{I}_{[l]}$$ where any input signal $u(t)$.

$$\exists \rho_{i,j} \in \mathbb{R} \ \text{ s.t. } \ g_i(s) \equiv \rho_{i,j} g_j(s), \ \forall i, j \in \mathcal{I}_{[l]}$$ where $g_i(s) := \frac{\mathscr{L}[x_i]}{\mathscr{L}[u]}$.

Basis Expansion via Positive Tri-diagonalization

[T. Ishizaki *et al.* CDC 2010]

Dynamical network (A, b)

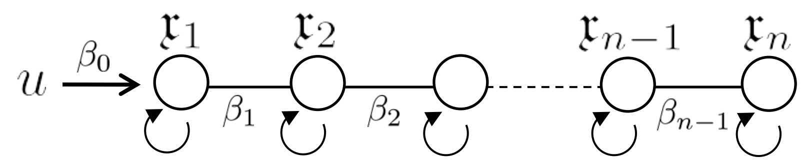


$$x = H\mathfrak{x}$$

$$H = \{h_{i,j}\} \in \mathbb{R}^{n \times n}$$

$$\dot{x} = Ax + bu, \quad \begin{cases} A = A^{\mathsf{T}} \in \mathbb{R}^{n\times n} \\ b \in \mathbb{R}^n \end{cases}$$

Positive tri-diagonal pair $(\mathfrak{A}, \mathfrak{b})$



$$\mathfrak{A} = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{bmatrix} \quad \mathfrak{b} = \begin{bmatrix} \beta_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

tri-diagonal

<u>basis expansion</u> $\quad g_i(s) = \sum_{j=1}^{n} h_{i,j}\mathfrak{g}_j(s)$

$$\mathfrak{g}_i(s) := \frac{\mathscr{L}[\mathfrak{x}_i]}{\mathscr{L}[u]}$$

DC-gain is maximal

$$\|\mathfrak{g}_i(s)\|_{\mathcal{H}_\infty} = \mathfrak{g}_i(0)$$

$$\|g_i(s) - \rho_{i,j}g_j(s)\|_{\mathcal{H}_\infty} = 0 \;\Leftrightarrow\; \mathrm{row}_i\left[H\mathrm{diag}(\mathfrak{g})\right] = \rho_{i,j}\,\mathrm{row}_j\left[H\mathrm{diag}(\mathfrak{g})\right]$$

where $\mathfrak{g} := -\mathfrak{A}^{-1}\mathfrak{b} = [\mathfrak{g}_1(0), \ldots, \mathfrak{g}_n(0)]^{\mathsf{T}}$

Aggregation of Reducible Clusters

Cluster reducibility: $\exists \rho_{i,j} \in \mathbb{R}$ s.t. $g_i(s) = \rho_{i,j} g_j(s), \ \forall i,j \in \mathcal{I}_{[l]}$

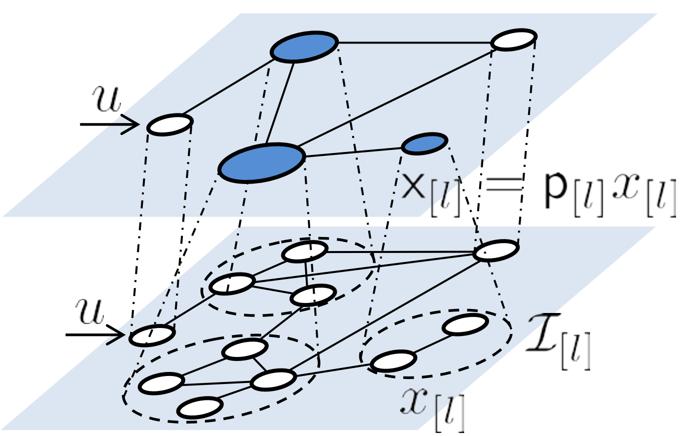
[Theorem] Let $\mathcal{I}_{[l]}$ be the index set of the l-th cluster.

The l-th cluster is reducible if and only if

$$\exists \rho_{i,j} \in \mathbb{R} \ \text{ s.t. } \ \mathrm{row}_i \left[H \mathrm{diag}\left(\mathfrak{g}\right)\right] = \rho_{i,j} \, \mathrm{row}_j \left[H \mathrm{diag}\left(\mathfrak{g}\right)\right], \ \forall i,j \in \mathcal{I}_{[l]}.$$

Furthermore, $g(s) = \mathsf{g}(s)$ holds for $\mathsf{p}_{[l]} = \frac{p_{[l]}}{\|p_{[l]}\|}, \ p_{[l]} = \{\rho_{i,j}\}_{j \in \mathcal{I}_{[l]}} \in \mathbb{R}^{1 \times |\mathcal{I}_{[l]}|}$.

P : unitary, i.e., $\mathsf{P}^{\mathsf{T}} = \mathsf{P}^{\dagger}$



Aggregated model $(\mathsf{P}A\mathsf{P}^{\mathsf{T}}, \mathsf{P}b)$

$$\mathsf{g}(s) = \mathsf{P}^{\mathsf{T}}\left(sI_{\Delta} - \mathsf{P}A\mathsf{P}^{\mathsf{T}}\right)^{-1}\mathsf{P}b$$

Dynamical network (A, b)

$$g(s) = (sI_n - A)^{-1} b$$

- Elimination of uncontrollable subspace with block-diagonally structured P

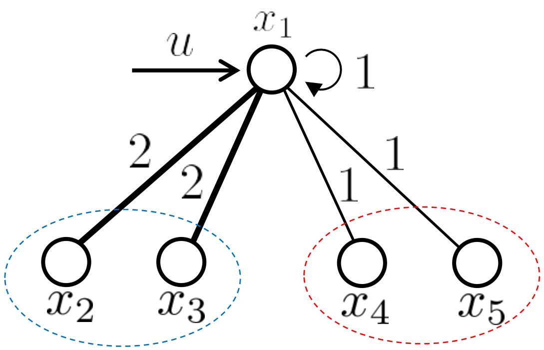
Simple Example

Reducibility:

$$\exists \rho_{i,j} \text{ s.t. } \mathrm{row}_i\left[H\mathrm{diag}(\mathfrak{g})\right] = \rho_{i,j}\, \mathrm{row}_j\left[H\mathrm{diag}(\mathfrak{g})\right]$$

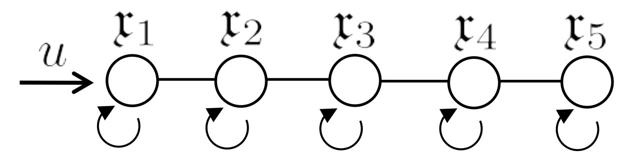
where $\mathfrak{g} = -\mathfrak{A}^{-1}\mathfrak{b}$

Given $g(s) = (sI_n - A)^{-1} b$



$$x = H\mathfrak{x}$$

positive tri-diagonal pair $(\mathfrak{A}, \mathfrak{b})$



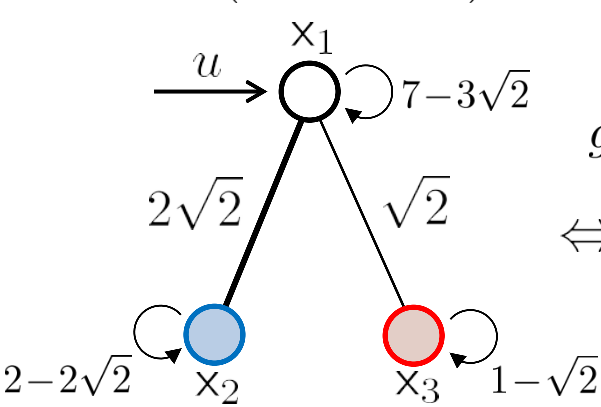
$$H\mathrm{diag}(\mathfrak{g}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1.2 & -0.2 & 0 & 0 \\ 0 & 1.2 & -0.2 & 0 & 0 \\ 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0.6 & 0.4 & 0 & 0 \end{bmatrix}$$

aggregation

$\mathsf{x} = \mathsf{P}x$

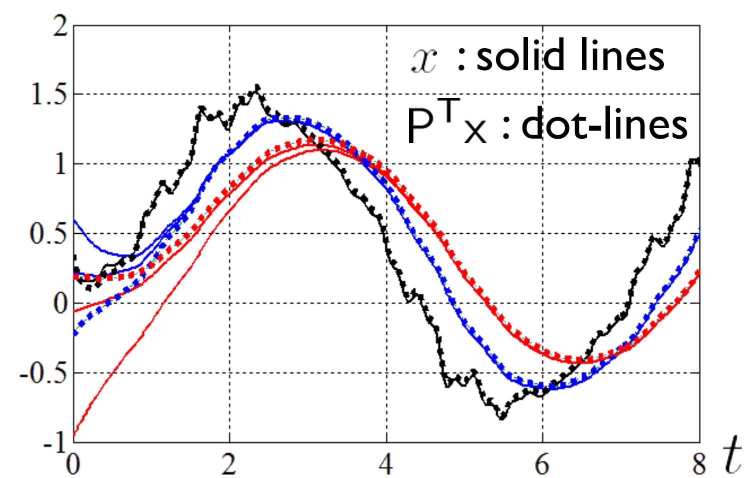
$$\mathsf{P} = \begin{bmatrix} 1 & & \\ & \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] & \\ & & \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \end{bmatrix}$$

$$\mathsf{g}(s) = \mathsf{P}^{\mathsf{T}}\left(sI_{\Delta} - \mathsf{P}A\mathsf{P}^{\mathsf{T}}\right)^{-1}\mathsf{P}b$$



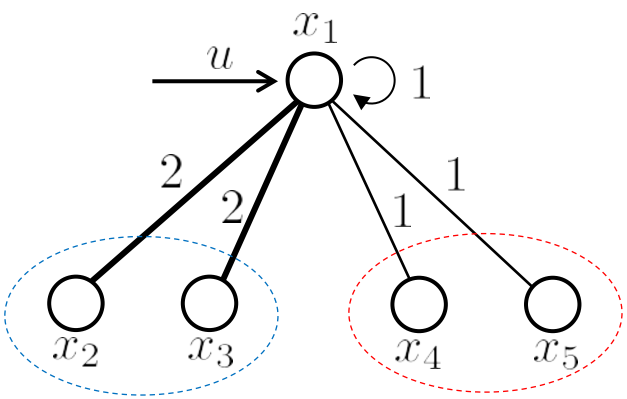
$g(s) = \mathsf{g}(s)$ holds

$\Leftrightarrow \mathsf{P}^{\mathsf{T}}\mathsf{x} \to x, \forall u, x(0), \mathsf{x}(0)$



Relaxation of Reducibility Condition

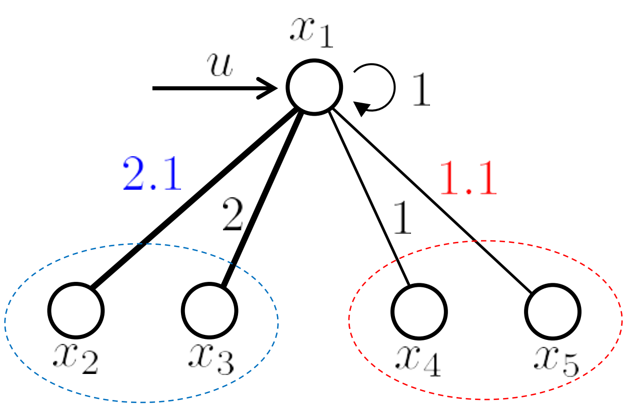
Dynamical network (A, b)



$$H\mathrm{diag}(\mathfrak{g}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1.2 & -0.2 & 0 & 0 \\ \hline 0 & 1.2 & -0.2 & 0 & 0 \\ \hline 0 & 0.6 & 0.4 & 0 & 0 \\ \hline 0 & 0.6 & 0.4 & 0 & 0 \end{bmatrix}$$

<u>exactly same</u> behavior

$$\mathrm{row}_i\left[H\mathrm{diag}(\mathfrak{g})\right] = \rho_{i,j}\,\mathrm{row}_j\left[H\mathrm{diag}(\mathfrak{g})\right]$$



$$H\mathrm{diag}(\mathfrak{g}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1.23 & -0.25 & 0.03 & -0.00 \\ \hline 0 & 1.17 & -0.14 & -0.03 & 0.01 \\ \hline 0 & 0.58 & 0.39 & 0.02 & 0.01 \\ \hline 0 & 0.64 & 0.38 & -0.01 & -0.01 \end{bmatrix}$$

<u>similar</u> behavior

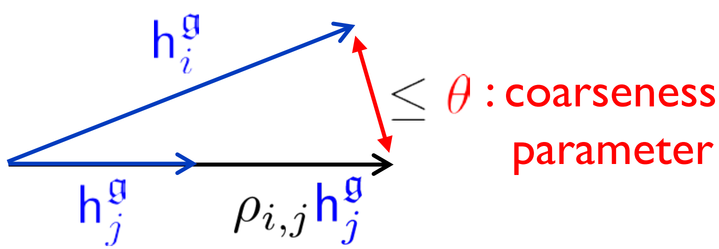
$$\mathrm{row}_i\left[H\mathrm{diag}(\mathfrak{g})\right] \simeq \rho_{i,j}\,\mathrm{row}_j\left[H\mathrm{diag}(\mathfrak{g})\right]$$

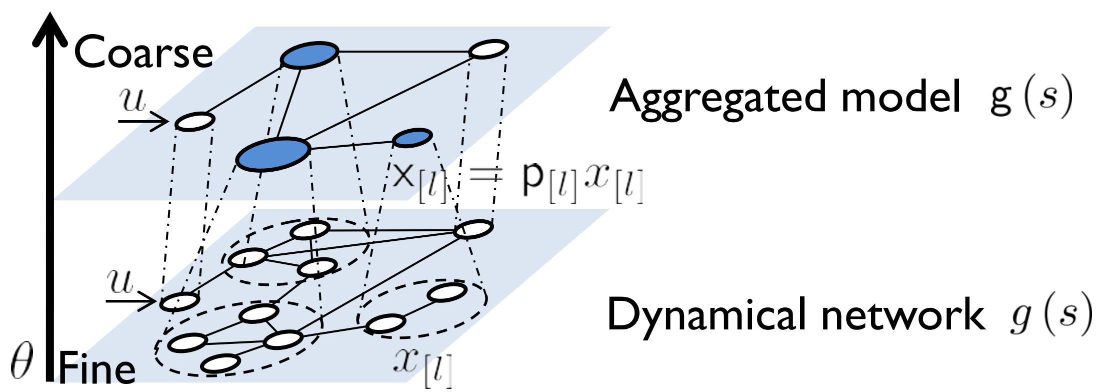
Aggregation of Weakly Reducible Clusters

Denote $\mathrm{row}_i\left[H\mathrm{diag}(\mathfrak{g})\right] = \mathsf{h}_i^{\mathfrak{g}}$ and define $p = \{p_i\} = (-A^{-1}b)^{\mathsf{T}} \in \mathbb{R}^{1\times n}$.

[Definition] θ-weakly Reducible Cluster ※ $\rho_{i,j} := p_i/p_j$

The cluster $\mathcal{I}_{[l]}$ is *θ-weakly reducible* if $\exists i \in \mathcal{I}_{[l]}$ s.t. $\left\|\mathsf{h}_i^{\mathfrak{g}} - \rho_{i,j}\mathsf{h}_j^{\mathfrak{g}}\right\| \le \theta, \ \forall j \in \mathcal{I}_{[l]}$.





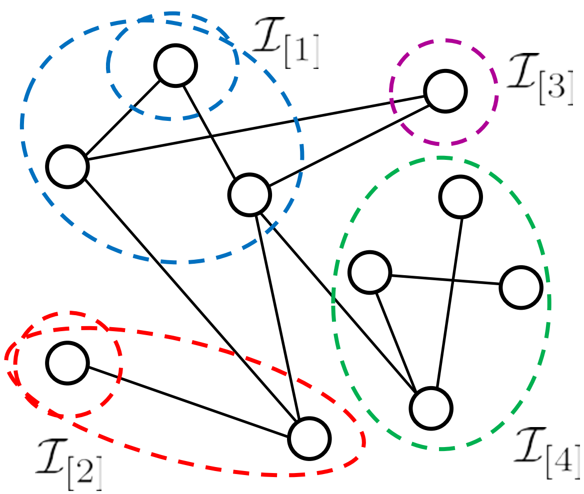
[Theorem] Let $p_{[l]} = \{\rho_{i,j}\}_{j\in\mathcal{I}_{[l]}} \in \mathbb{R}^{1\times|\mathcal{I}_{[l]}|}$.

Suppose all clusters are θ-weakly reducible, and take $\mathsf{p}_{[l]} = \dfrac{p_{[l]}}{\|p_{[l]}\|} \in \mathbb{R}^{1\times|\mathcal{I}_{[l]}|}$.

Then, $g(0) = \mathsf{g}(0)$, $\|g(s) - \mathsf{g}(s)\|_{\mathcal{H}_\infty} \le \alpha\theta$ hold for an α determined by A .

Algorithm to Construct Weakly Reducible Cluster Set

Given network



- Give $\theta \in \mathbb{R}_+$, Initialize $\{\mathcal{I}_{[l]}\}_{l\in\mathbb{L}} = \emptyset,\ \mathbb{L} = \emptyset,\ l = 0$

While $\{\mathcal{I}_{[l]}\}_{l\in\mathbb{L}} \neq \{1,\ldots,n\}$

- $l{+}{+},\ \mathbb{L} \leftarrow \{\mathbb{L}, l\}$
- Choose $i \in \{1,\ldots,n\}/\{\mathcal{I}_{[l]}\}_{l\in\mathbb{L}}$, Set $\mathcal{I}_{[l]} = \{i\}$
- For all $j \in \{1,\ldots,n\}/\{\mathcal{I}_{[l]}\}_{l\in\mathbb{L}}$,

  if (i,j) satisfies $(*)$, then $\mathcal{I}_{[l]} \leftarrow \{\mathcal{I}_{[l]}, j\}$

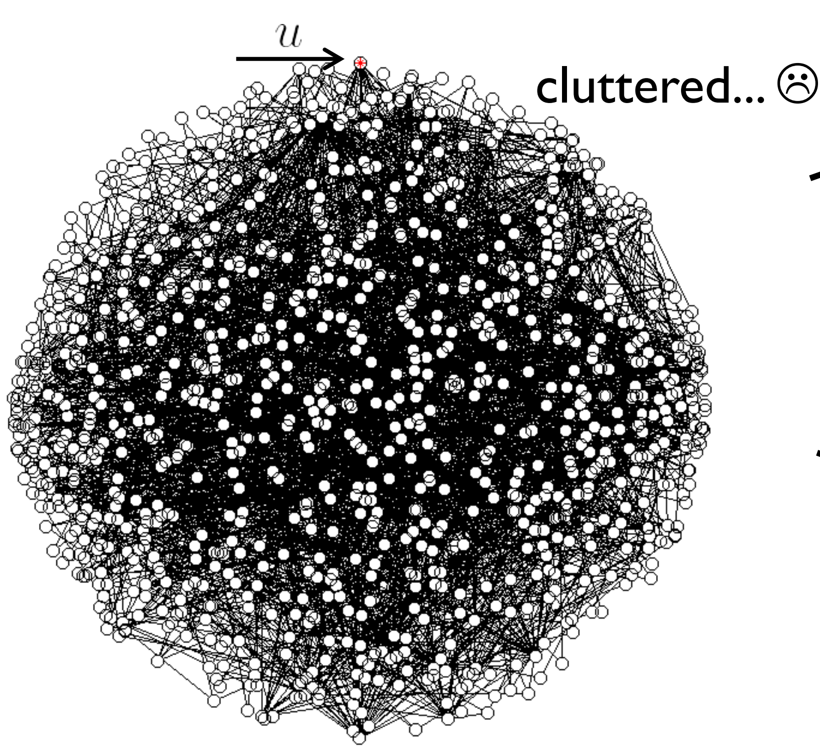
θ-weak reducibility condition:

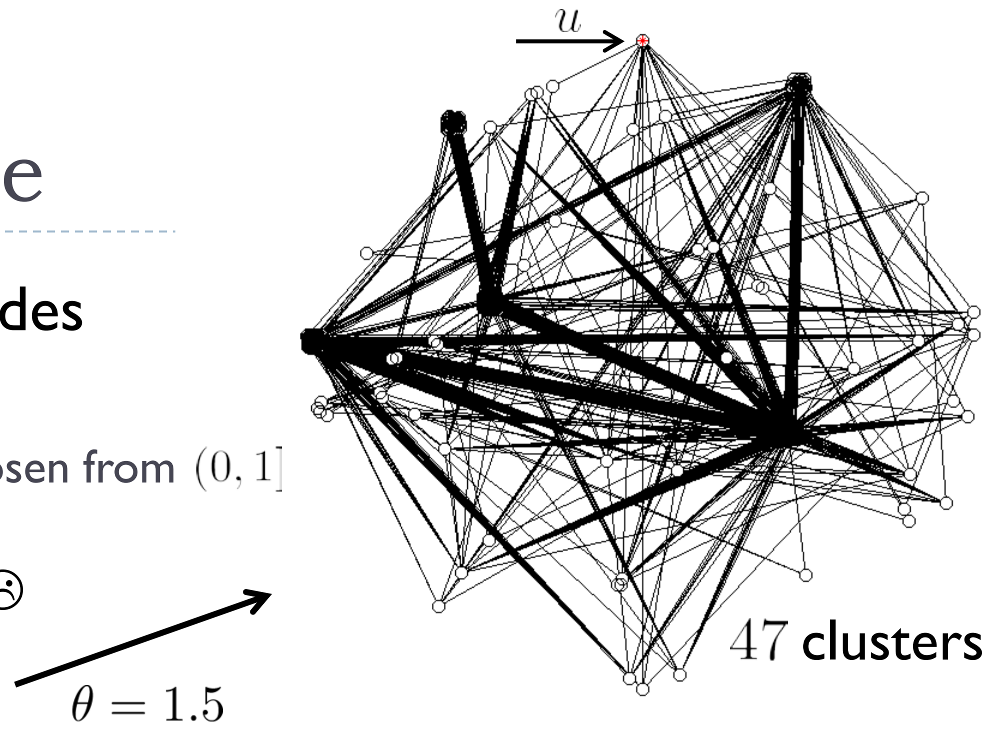
$$\left\| \mathsf{h}_i^{\mathfrak{g}} - \rho_{i,j}\mathsf{h}_j^{\mathfrak{g}} \right\| \leq \theta \quad \cdots\cdots \quad (*)$$

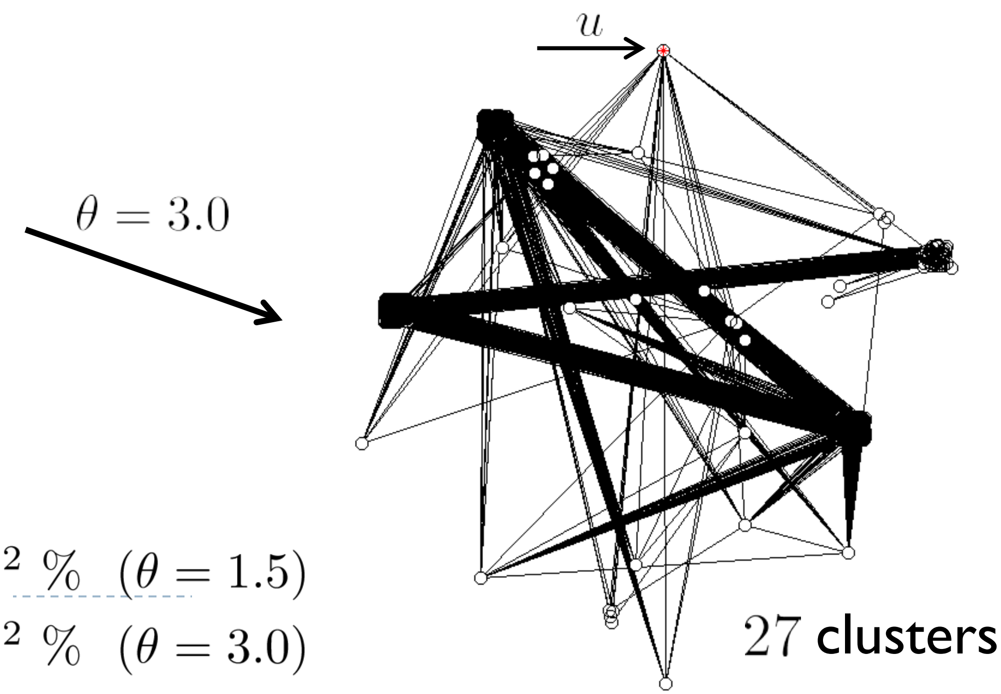
where $\mathsf{h}_i^{\mathfrak{g}} := \mathrm{row}_i\left[H\mathrm{diag}\left(\mathfrak{g}\right)\right]$ and $\rho_{i,j} = p_i/p_j$

Numerical Example

- Holme-Kim model of 1000 nodes
  - a famous complex network model
  - non-zero edge weight is randomly chosen from $(0, 1]$







- $$\frac{\|g - \mathrm{g}\|_{\mathcal{H}_\infty}}{\|g\|_{\mathcal{H}_\infty}} = \begin{cases} 5.93 \times 10^{-2}\ \% & (\theta = 1.5) \\ 9.09 \times 10^{-2}\ \% & (\theta = 3.0) \end{cases}$$

Remarks on $\mathcal{H}_\infty$-aggregation

- Similar aggregation approach admits to multi-input systems
  - positive tri-diagonalization with respect to each input signal
- Aggregation of positive networks preserves the positivity
  - positivity: non-negativity of off-diagonal entries of A and entries of b
- Generalization to *positive directed networks* is also possible
  - asymmetric A with non-negative off-diagonal entries
  - tri-diagonalization is replaced with Hessenberg transformation
  - however, higher computational costs are to be required

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$\mathcal{H}_2$-aggregation based on Controllability Gramian

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$\mathcal{H}_2$-norm: $\|\Sigma\|_{\mathcal{H}_2} := \left(\int_0^\infty \|Ce^{At}B\|_F^2 \, dt \right)^{\frac{1}{2}}$

Positive semi-definite solution Φ of Lyapunov equation $A\Phi + \Phi A^{\mathsf{T}} + BB^{\mathsf{T}} = 0$

Φ : controllability gramian

- $\|\Sigma\|_{\mathcal{H}_2} = \left(\mathrm{trace}[C\Phi C^{\mathsf{T}}]\right)^{\frac{1}{2}}$
- Φ is non-singular $\Leftrightarrow$ (A, B) is controllable

Φ relates to $\mathcal{H}_2$ -norm & controllability

[Theorem]

The cluster $\mathcal{I}_{[l]}$ is reducible if and only if

$$\exists \rho_{i,j} \in \mathbb{R} \ \text{ s.t. } \ \mathrm{row}_i[\Phi^{\frac{1}{2}}] = \rho_{i,j} \, \mathrm{row}_j[\Phi^{\frac{1}{2}}], \ \forall i, j \in \mathcal{I}_{[l]}.$$

Aggregation of Weakly Reducible Clusters

Denote $\mathrm{row}_i[\Phi^{\frac{1}{2}}] = \phi_i$ and define $p = \{p_i\} = (-A^{-1}b)^{\mathsf{T}} \in \mathbb{R}^{1\times n}$.

[Definition] θ**-weakly Reducible Cluster**

The cluster $\mathcal{I}_{[l]}$ is *θ-weakly reducible* if $\exists i \in \mathcal{I}_{[l]}$ s.t. $\|\phi_i - \rho_{i,j}\phi_j\| \leq \theta, \quad \forall j \in \mathcal{I}_{[l]}$.

[Theorem] Let $p_{[l]} = \{\rho_{i,j}\}_{j\in\mathcal{I}_{[l]}} \in \mathbb{R}^{1\times|\mathcal{I}_{[l]}|}, \quad \rho_{i,j} := p_i/p_j$.

Suppose all clusters are θ-weakly reducible, and take $\mathsf{p}_{[l]} = \dfrac{p_{[l]}}{\|p_{[l]}\|} \in \mathbb{R}^{1\times|\mathcal{I}_{[l]}|}$.

Then, $g(0) = \mathsf{g}(0)$, $\|g(s) - \mathsf{g}(s)\|_{\mathcal{H}_2} \leq \alpha\theta$ hold for a positive constant α.

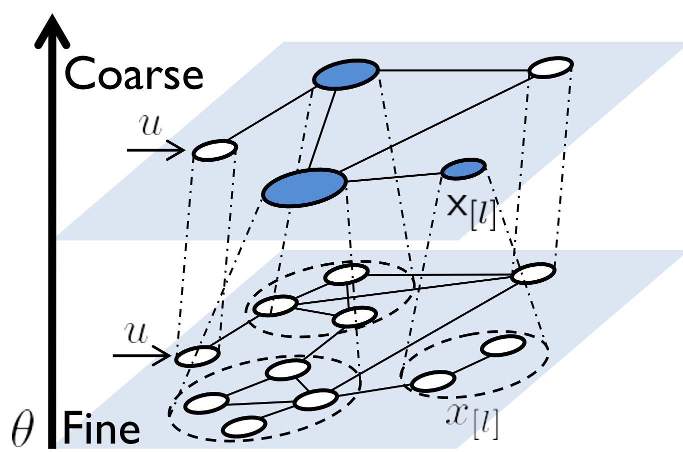
Remarks on $\mathcal{H}_2$-aggregation

- Lyapnov equation $A\Phi + \Phi A^{\mathsf{T}} + BB^{\mathsf{T}} = 0$ is identical regardless of
  - single- or multi-input systems
  - moreover, regardless of symmetry of A

- Generalization to *positive directed networks* is also possible
  - preserving network topology, stability, and positivity
  - that to systems with arbitrary stable A is difficult
    - not only error evaluation, but also guaranteeing stability of reduced model

Summary

- Model reduction based on state aggregation is proposed
  - preserving network structure
  - aggregation of *reducible* clusters
    - characterization via positive tri-diagonalization leads to $\mathcal{H}_\infty$-aggregation
    - characterization via controllability gramian leads to $\mathcal{H}_2$-aggregation
- Aggregation techniques have ability to preserve other properties
  - system positivity, second order (oscillator) structure, etc…



Aggregated model $(\mathsf{P}A\mathsf{P}^\mathsf{T}, \mathsf{P}b)$

$$\mathsf{g}(s) = \mathsf{P}^\mathsf{T}\left(sI_\Delta - \mathsf{P}A\mathsf{P}^\mathsf{T}\right)^{-1}\mathsf{P}b$$

Dynamical network (A, b)

$$g(s) = (sI_n - A)^{-1}\, b$$