Hierarchical Decentralized Observer Design for Linearly Coupled Network Systems

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***Abstract*— This paper proposes a novel type of decentralized observers for a network system, where identical linear subsystems are interconnected. For this system, we derive a state-space model with block-triangular structure, in which the dynamics of the interaction among the subsystems and the dynamics of each subsystem are decoupled. Based on the decoupled model, we design a hierarchical decentralized observer, where a kind of centralized observer estimates coarse information on interaction among the subsystems and a decentralized observer estimates the state of each subsystem. Furthermore, we derive a necessary and sufficient condition of the observability for the decentralized estimation under applying the hierarchical decentralized observer.**

I. Introduction

For large-scale network systems, various methods of distributed/decentralized control have been intensively developed over the past few decades, in order to overcome difficulties in heavy computation costs. One feature of these methods is that the structure of feedback gain matrices is restricted to reduce the amount of communication. More precisely, the gains of the distributed controllers are structured accordingly to a communication graph, which is often same as the interconnection topology of systems [1], [2], [3]. This structure only allows to feed back the state of a subsystem to the adjacent ones. On the other hand, the gains of the decentralized controllers are block-diagonally structured [4], [5]. This allows a local feedback, where the state of a subsystem is fed back only to itself. Furthermore, as the dual notion, much attention has been also paid on distributed/decentralized observation [6], [7], [8], [9]. Many conventional results such as decentralized Kalman filter have been developed by the 1990s. In addition, the observability problem for large-scale network systems has been extensively addressed within a few years [10], [11], [12].

As one of large-scale network systems, we focus on systems where identical linear subsystems are interconnected. In this paper, we propose a sort of decentralized observer for this network system. The proposed observer, which we call a hierarchical decentralized observer, is composed of two kinds of observers; one is a global observer, which coarsely estimates interaction among the subsystems (*coarse information*), and the other is a set of local observers, which exactly estimates the internal state on each subsystem (*fine information*). This composition means that the signal transduction among the whole system is decoupled based on its fineness. With this, the feedback gain matrix is to be block-diagonally structured. Thus, the architecture of the proposed observer is essentially different from that of the existing decentralized/distributed observers.

The key idea to design such a hierarchical observer is introducing a state-space model with block-triangular structure, in which the dynamics of the interaction among the subsystems and the dynamics on each subsystem are decoupled. This model is derived from the viewpoints of overlapping expansion of the original state-space, as well as suitable contraction of the expanded state-space. Consequently, this state-space model has generally a different dimension from that of the original system, and realizes the decoupling of the dynamics. These features are similar to those of overlapping models introduced in, e.g., [7], [13], [14]. However, the dynamics of subsystems is not completely decoupled in those models. Furthermore, we analyze the observability of the state-space model under applying the hierarchical decentralized observer. It turns out that our result is a more general version of the result in [15], although we adopt a different approach to derive a necessary and sufficient condition for the observability.

This paper is organized as follows: In Section II-A, we describe network systems to be studied and formulate a problem of a hierarchical decentralized observer design. Then, in Section II-B, we solve the design problem by deriving a state-space model with block-triangular structure, and we investigate the observability condition of this model in Section II-C. In Section III, we verify the effectiveness of the proposed observer by a numerical example, and Section IV concludes this paper.

NOTATION: For a vector v and matrices $M=\{m_{i,j}\}$ and N, the following notation is used in this paper:

$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
I_n	the unit matrix of the size $n\times n$
e_k^n	the k-th column vector of I_n
$M\otimes N$	the Kronecker product of M and N, namely $\{m_{i,j}N\}$
$\mathfrak{R}(M)$	the range space of M
$\mathfrak{N}(M)$	the null space of M
$\mathfrak{E}_\lambda(M)$	the set $\{v \mid Mv=\lambda v, v\neq 0\}$
$\Lambda(M)$	the set $\{\lambda \mid Mv=\lambda v, v\neq 0\}$

Let $\mathcal{I}$ be the set of integers. The matrix $\mathrm{diag}(M_i)_{i\in\mathcal{I}}$ denotes the block-diagonal matrix whose block-diagonal elements are given by M_i for $i\in\mathcal{I}$. Furthermore, $|\mathcal{I}|$ denotes the

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cardinality of $\mathcal{I}$, and $e^n_{\mathcal{I}} \in \mathbb{R}^{n\times|\mathcal{I}|}$ denotes the matrix whose column vectors are composed of e^n_k for $k \in \mathcal{I}$ (in some order of k), i.e., $e^n_{\mathcal{I}} = \left[e^n_{k_1}, \ldots, e^n_{k_m}\right] \in \mathbb{R}^{n\times m}$ for $\mathcal{I} = \{k_1, \ldots, k_m\}$.

II. Hierarchical Decentralized Observer Design

A. Problem Formulation

In this paper, we deal with the following coupled linear systems: Suppose that N identical subsystems

$$\Sigma_i : \begin{cases} \dot{x}_i = a_I x_i + b_I u_i, \quad a_I \in \mathbb{R}^{n\times n},\ b_I \in \mathbb{R}^{n\times m} \\ w_i = c_I x_i, \quad c_I \in \mathbb{R}^{m\times n} \end{cases} \tag{1}$$

with the state $x_i \in \mathbb{R}^n$ for $i \in \{1, \ldots, N\}$ are interconnected by the linear coupling

$$u_i = \sum_{k=1}^{N} \gamma_{i,k} w_k, \tag{2}$$

for which the diagonalizable $\Gamma = \{\gamma_{i,j}\} \in \mathbb{R}^{N\times N}$ represents the interconnection structure of the subsystems. Then, by organizing the states as $x = [x_1^\mathsf{T}, \ldots, x_N^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{Nn}$, the state equation of the above network system is formed into

$$\begin{aligned} &\dot{x} = Ax, \quad x(0) = x_0 \\ &A = I_N \otimes a_I + \Gamma \otimes b_I c_I \in \mathbb{R}^{Nn\times Nn}, \end{aligned} \tag{3}$$

where $b_I c_I \neq 0$ is assumed without loss of generality. In what follows, we call x_i in (1) "the internal state on the i-th subsystem".

Next, in order to formulate a hierarchical decentralized (i.e., a kind of distributed) state estimation problem, we define two kinds of measured outputs, which we call a *global* output and a *local* output. Denote the global output by $y^G \in \mathbb{R}^{Pp_G}$ and the local output by $y^L_{\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|p_L}$, where $\mathcal{I} \subseteq \{1, \ldots, N\}$ represents an index set of the subsystems. Then, the output equation is defined by

$$\begin{aligned} &\begin{bmatrix} y^G \\ y^L_{\mathcal{I}} \end{bmatrix} = Cx \\ &C = \begin{bmatrix} \Psi \otimes c_G \\ (e^N_{\mathcal{I}})^\mathsf{T} \otimes c_L \end{bmatrix} \in \mathbb{R}^{(Pp_G + |\mathcal{I}|p_L)\times Nn} \end{aligned} \tag{4}$$

where $\Psi \in \mathbb{R}^{P\times N}$, $c_G \in \mathbb{R}^{p_G\times n}$ and $c_L \in \mathbb{R}^{p_L\times n}$. The system (A, C) is a generalization of the system dealt with in [15], [16], where the identical SISO subsystem is described by a generalized frequency variable.

In order to explain (4), let us introduce two outputs of the i-th subsystem denoted by $y^G_i = c_G x_i$ and $y^L_i = c_L x_i$. By using this, the global output y^G is rewritten by the linear combination

$$y^G = \sum_{k=1}^{N} \psi_k \otimes y^G_k$$

where $\psi_k \in \mathbb{R}^P$ denotes the k-th column vector of Ψ. It may, for example, represent the available *(weighted) average output* of subsystems. Furthermore, the local output $y^L_{\mathcal{I}}$ is rewritten by $y^L_{\mathcal{I}} = [y^L_i]_{i\in\mathcal{I}}$ (i.e., the vector composed of y^L_i for $i \in \mathcal{I}$). The interconnection structure of the network system (A, C) is depicted in Fig. 1.

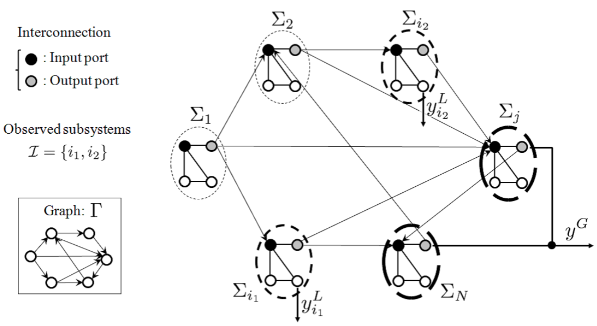

Fig. 1. Linearly Coupled Network System.

In what follows, we formulate a hierarchical decentralized (i.e., a kind of distributed) state estimation problem for the network system (A, C). Here, the decentralized state estimation means that the state x_i is estimated by only using the local output y^L_i for each $i \in \mathcal{I}$. However, such decentralized estimation is generally difficult because input-output information (interaction) among the subsystems (i.e., u_i in (2)) is unknown.

On the other hand, it is observed that the interaction among subsystems is relatively *coarser* (more contracted) than the interaction among the states on each subsystem. For example, $c_I = [1, \ldots, 1]$ in (1) means that the centroid of x_i is only involved in the interaction among the subsystems, while each x_i has the n-dimensional dynamics interacting via a_I. Actually, such a situation appears in, e.g., a kind of biological networks, where a protein interacts with the others by their averaged state while each of proteins has some determinate dynamics [17]. This fact suggests a possibility that the states x_i are estimated by decoupling into *coarse interaction* among subsystems described by (2) and *fine interaction* within the individual subsystems described by (1).

In order to realize this decoupling, we consider estimating the interaction (2) by using an extra output, which we call the global output y^G. However in general, it is not clear that what measured output y^G is desirable for the estimation of the interaction. Thus, we first introduce the following state-space representation of the system for hierarchical decentralized estimation, and then, we discuss how to derive such a representation.

Definition 1: Consider the network system (A, C) in (3) and (4), and let $\mathcal{I}$ be a given index set. The state-space representation $(\mathcal{A}, \mathcal{C})$ of (A, C) is said to be a *hierarchical state-space representation* if $\mathcal{A}$ and $\mathcal{C}$ are in the form of

$$\begin{aligned} &\mathcal{A} = \begin{bmatrix} \alpha_z & 0 \\ \beta_z & I_{|\mathcal{I}|} \otimes a_I \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} \gamma_z & 0 \\ 0 & I_{|\mathcal{I}|} \otimes c_L \end{bmatrix} \\ &\alpha_z \in \mathbb{R}^{n_z\times n_z},\ \beta_z \in \mathbb{R}^{|\mathcal{I}|n\times n_z},\ \gamma_z \in \mathbb{R}^{Pp_G\times n_z} \end{aligned} \tag{5}$$

and satisfy

$$\mathcal{C}\mathcal{X}(t) \equiv \begin{bmatrix} y^G(t) \\ y^L_{\mathcal{I}}(t) \end{bmatrix}, \ x_{\mathcal{I}}(t) \equiv \left\{(e^N_{\mathcal{I}})^\mathsf{T} \otimes I_n\right\} x(t), \ \forall t \tag{6}$$

where $\mathcal{X} = [z^\mathsf{T}, x_\mathcal{I}^\mathsf{T}]^\mathsf{T}$ for $z \in \mathbb{R}^{n_z}$ and $x_\mathcal{I} \in \mathbb{R}^{|\mathcal{I}|n}$ obeys

$$\begin{aligned} &\dot{\mathcal{X}} = \mathcal{A}\mathcal{X}, \;\; \mathcal{X}(0) \in \mathbb{X} \\ &\mathbb{X} := \left\{ \mathcal{X} \;\middle|\; \mathcal{X} \in \left[\begin{array}{c} \sigma_z x \\ \{(e_\mathcal{I}^N)^\mathsf{T} \otimes I_n\}\, x \end{array} \right],\; x \in \mathbb{R}^{Nn} \right\} \end{aligned} \tag{7}$$

for some matrix $\sigma_z \in \mathbb{R}^{n_z \times Nn}$.

In this hierarchical state-space representation (hereafter denoted as **HSS**-representation), z represents the state of interaction among the subsystems, and $x_\mathcal{I}$ represents the state to be estimated. Based on the **HSS**-representation, we also define a hierarchical decentralized observer composed of two kinds of the observer O_z and o_i, where O_z estimates z from y^G, and o_i individually estimates x_i from y_i^L:

Definition 2: Consider the **HSS**-representation $(\mathcal{A}, \mathcal{C})$ in (5) and (7). Let H and $h_\mathcal{I} = \mathrm{diag}(h_i)_{i\in\mathcal{I}}$ be matrices, and

$$\begin{aligned} O_z \;&:\; \dot{\hat{z}} = (\alpha_z - H\gamma_z)\,\hat{z} + Hy^G \\ \{o_i\}_{i\in\mathcal{I}} \;&:\; \dot{\hat{x}}_\mathcal{I} = \left\{ I_{|\mathcal{I}|} \otimes a_I - h_\mathcal{I}\left(I_{|\mathcal{I}|} \otimes c_L\right)\right\} \hat{x}_\mathcal{I} \\ &\qquad\qquad + \beta_z \hat{z} + h_\mathcal{I} y_\mathcal{I}^L. \end{aligned} \tag{8}$$

Then, $O(O_z, \{o_i\}_{i\in\mathcal{I}})$ is said to be a *hierarchical decentralized observer* if $\lim_{t\to\infty} (\hat{\mathcal{X}}(t) - \mathcal{X}(t)) = 0$ for all $\hat{\mathcal{X}}(0) \in \mathbb{R}^{(n_z + |\mathcal{I}|n)}$ and $\mathcal{X}(0) \in \mathbb{X}$, where $\hat{\mathcal{X}} = [\hat{z}^\mathsf{T}, \hat{x}_\mathcal{I}^\mathsf{T}]^\mathsf{T}$.

This observer can be straightforwardly constructed by using α_z, β_z and γ_z in (5). Note that in this hierarchical decentralized observer (hereafter denoted as a **HD**-observer), the global output y^G is fed back only to the observer O_z, while the local output $y_\mathcal{I}^L$ of the subsystems $\{\Sigma_i\}_{i\in\mathcal{I}}$ is fed back only to the observer $\{o_i\}_{i\in\mathcal{I}}$. Furthermore, from

$$\begin{aligned} &\hat{\mathcal{X}}(t) - \mathcal{X}(t) \\ &\quad = \exp\left\{(\mathcal{A} - \mathrm{diag}(H, h_\mathcal{I})\,\mathcal{C})\,t\right\} \left(\hat{\mathcal{X}}(0) - \mathcal{X}(0)\right) \end{aligned} \tag{9}$$

we can easily prove the following result:

Proposition 1: The observer (8) is the **HD**-observer if and only if both $(\alpha_z - H\gamma_z)$ and $(\alpha_I - h_i c_L)$ for all $i \in \mathcal{I}$ are Hurwitz.

Proposition 1 shows that all feedback gains H and h_i for $i \in \mathcal{I}$ can be independently designed. This further indicates that if we once design the global observer, we can easily plug-in/out the other local observers to estimate the state on subsystems. Moreover, the existence of the **HD**-observer is guaranteed by the observability of (α_z, γ_z) and (a_I, c_L), which represent the dynamics of z and the subsystem, respectively. Thus, the fundamental problems that we should consider are

- the derivation of a **HSS**-representation $(\mathcal{A}, \mathcal{C})$
- the observability analysis of (α_z, γ_z) and (a_I, c_L).

These problems are investigated in Section II-B and II-C below. In the rest of this subsection, we give an illustrative example:

Example: Let $\mathcal{I} = \{1, 2\}$ in (5). Then, the dynamics of $z \in \mathbb{R}^{n_z}$ and $x_\mathcal{I} = [x_1^\mathsf{T}, x_2^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{2n}$ in the **HSS**-

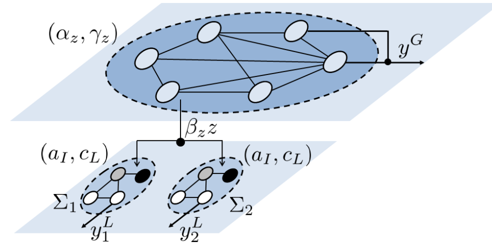

Fig. 2. Hierarchical State-Space Representation.

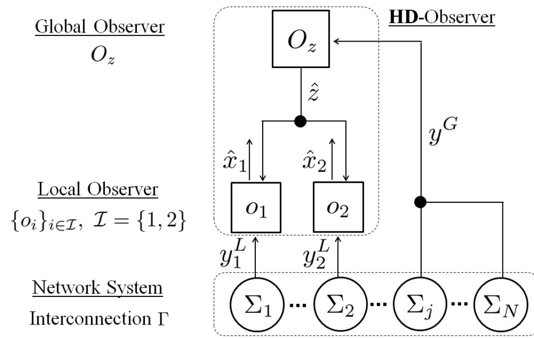

Fig. 3. Hierarchical Decentralized Observer.

representation stands for

$$\left\{ \begin{array}{l} \dot{z} = \alpha_z z \\ y^G = \gamma_z z \end{array} \right., \left\{ \begin{array}{l} \left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array} \right] = \left[\begin{array}{cc} a_I & \\ & a_I \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] + \beta_z z \\ \left[\begin{array}{c} y_1^L \\ y_2^L \end{array} \right] = \left[\begin{array}{cc} c_L & \\ & c_L \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right]. \end{array} \right.$$

From these equations, we see that the internal states on the subsystems are determined by the individual system $\dot{x}_i = a_I x_i$ for $i = 1, 2$, and the interaction term $\beta_z z$. Thus, if the state z is estimated by the observer O_z, each x_i can be estimated in a decentralized manner by each observer o_i. This is the fundamental architecture of the proposed observer; see Fig. 3. The architecture is essentially different from that of the existing decentralized/distributed observers, where the structure of feedback gain matrices is restricted [6], [7], [8], [9].

Remark 1: The dimension is $(n_z + |\mathcal{I}|\, n)$ for the **HSS**-representation $(\mathcal{A}, \mathcal{C})$ and Nn for the original network system (A, C). These are different in general. In other words, $(\mathcal{A}, \mathcal{C})$ is derived from some contraction of an *overlapped* state-space of x, instead of the general coordinate transformation of (A, C); see Theorem 1 below for the details.

B. Derivation of Hierarchical State-Space Representation

In this subsection, we derive a **HSS**-representation $(\mathcal{A}, \mathcal{C})$ associated with the network system (A, C). The following theorem derives the **HSS**-representation by projecting Γ and the subsystem (a_I, b_I, c_I) onto a certain observable subspace. The observability matrix is denoted by

$$\mathcal{O}_n(A, C) = \left[\begin{array}{c} C \\ CA \\ \vdots \\ CA^{n-1} \end{array} \right].$$

Theorem 1: Consider the network system (A, C) in (3) and (4). Define

$$\nu := \mathrm{rank}\,(\Gamma)\,, \quad \mu := \mathrm{rank}\,(\mathcal{O}_n\,(a_I, b_I c_I))\,.$$

Let $\mathsf{N} \in \mathbb{R}^{N\times\nu}$ and $\mathsf{M} \in \mathbb{R}^{n\times\mu}$ be matrices such that $\mathsf{N}^\mathsf{T}\mathsf{N} = I_\nu$, $\mathsf{M}^\mathsf{T}\mathsf{M} = I_\mu$ and

$$\mathfrak{R}\,(\mathsf{N}) = \mathfrak{R}\left(\Gamma^\mathsf{T}\right),\ \mathfrak{R}\,(\mathsf{M}) = \mathfrak{R}\left(\mathcal{O}_n\,(a_I, b_I c_I)^\mathsf{T}\right). \quad (10)$$

If $\Psi \in \mathbb{R}^{P\times N}$ and $c_G \in \mathbb{R}^{p_G\times n}$ in (4) satisfy

$$\mathfrak{R}\left(\Psi^\mathsf{T}\right) \subseteq \mathfrak{R}\left(\Gamma^\mathsf{T}\right),\ \ \mathfrak{R}\left(c_G^\mathsf{T}\right) \subseteq \mathfrak{R}\left(\mathcal{O}_n\,(a_I, b_I c_I)^\mathsf{T}\right), \quad (11)$$

then the **HSS**-representation $(\mathcal{A}, \mathcal{C})$ associated with (A, C) is given by

$$\begin{cases} \alpha_z = I_\nu \otimes a_z + \Gamma_\nu \otimes b_z c_z \in \mathbb{R}^{\nu\mu\times\nu\mu} \\ \gamma_z = \Psi\mathsf{N} \otimes c_G\mathsf{M} \in \mathbb{R}^{Pp_G\times\nu\mu} \\ \beta_z = (e_\mathcal{I}^N)^\mathsf{T}\Gamma\mathsf{N} \otimes b_I c_z \in \mathbb{R}^{|\mathcal{I}|n\times\nu\mu} \\ \sigma_z = \mathsf{N}^\mathsf{T} \otimes \mathsf{M}^\mathsf{T} \in \mathbb{R}^{Nn\times\nu\mu} \end{cases} \quad (12)$$

for (5) and (7), where

$$\begin{aligned} &\Gamma_\nu := \mathsf{N}^\mathsf{T}\Gamma\mathsf{N} \in \mathbb{R}^{\nu\times\nu}, \quad a_z := \mathsf{M}^\mathsf{T} a_I \mathsf{M} \in \mathbb{R}^{\mu\times\mu}, \quad (13) \\ &b_z := \mathsf{M}^\mathsf{T} b_I \in \mathbb{R}^{\mu\times m}, \quad c_z := c_I\mathsf{M} \in \mathbb{R}^{m\times\mu}. \end{aligned}$$

Proof: Consider the redundant representation of (A, C) as

$$\begin{aligned} \begin{bmatrix} \dot{x}_G \\ \dot{x}_L \end{bmatrix} &= \begin{bmatrix} A & 0 \\ \Gamma \otimes b_I c_I & I_N \otimes a_I \end{bmatrix} \begin{bmatrix} x_G \\ x_L \end{bmatrix} \quad (14) \\ \begin{bmatrix} y^G \\ y^L_\mathcal{I} \end{bmatrix} &= \begin{bmatrix} \Psi \otimes c_G & \\ & (e_\mathcal{I}^N)^\mathsf{T} \otimes c_L \end{bmatrix} \begin{bmatrix} x_G \\ x_L \end{bmatrix} \end{aligned}$$

for $x_G\,(0) = x_L\,(0) = x_0$. Note that

$$\Gamma\mathsf{N}\mathsf{N}^\mathsf{T} = \Gamma, \quad \mathsf{M}^\mathsf{T} a_I = a_z \mathsf{M}^\mathsf{T}, \quad b_I c_I \mathsf{M}\mathsf{M}^\mathsf{T} = b_I c_I$$

follows from the properties of the invariant sub-space. Moreover, from the supposition (11)

$$\Psi\mathsf{N}\mathsf{N}^\mathsf{T} = \Psi, \quad c_G\mathsf{M}\mathsf{M}^\mathsf{T} = c_G$$

holds. Here, taking

$$\begin{bmatrix} z \\ x_\mathcal{I} \end{bmatrix} := L^\mathsf{T} \begin{bmatrix} x_G \\ x_L \end{bmatrix},\ L = \begin{bmatrix} \mathsf{N} \otimes \mathsf{M} & \\ & e_\mathcal{I}^N \otimes I_n \end{bmatrix}$$

and multiplying (14) by L^T from the left, we have

$$\begin{aligned} \begin{bmatrix} \dot{z} \\ \dot{x}_\mathcal{I} \end{bmatrix} &= \begin{bmatrix} \alpha_z & 0 \\ \beta_z & I_{|\mathcal{I}|} \otimes a_I \end{bmatrix} \begin{bmatrix} z \\ x_\mathcal{I} \end{bmatrix} \quad (15) \\ \begin{bmatrix} y^G \\ y^L_\mathcal{I} \end{bmatrix} &= \begin{bmatrix} \Psi\mathsf{N} \otimes c_G\mathsf{M} & \\ & I_{|\mathcal{I}|} \otimes c_L \end{bmatrix} \begin{bmatrix} z \\ x_\mathcal{I} \end{bmatrix}. \end{aligned}$$

Finally, by taking $\mathcal{X}\,(0) := L[x_G^\mathsf{T}(0), x_L^\mathsf{T}(0)]^\mathsf{T}$ the result follows. ∎

As shown in Theorem 1, Γ_ν is obtained by projecting Γ onto the range space of Γ^T, and (a_z, b_z, c_z) is obtained by projecting the subsystem (a_I, b_I, c_I) onto the observable sub-space of $(a_I, b_I c_I)$. Furthermore, (11) characterizes the condition that the global output y^G is exploited to estimate z. Note that even though the dynamics of z is obtained by eliminating the unobservable sub-spaces of (I_N, Γ) and $(a_I, b_I c_I)$, this does not mean $(\mathcal{A}, \mathcal{C})$ is observable; see Section II-C for the observability analysis.

Remark 2: The dimension $\nu\mu$ of the state z in the **HSS**-representation $(\mathcal{A}, \mathcal{C})$ quickly increases as the rank of Γ and the dimension of the observable sub-space of $(a_I, b_I c_I)$ become larger. From the model reduction perspective, the use of lower dimensional approximation of the state-space of z is more desirable instead of the *exact* system. This topic will be reported in the future.

C. Observability Criteria for Hierarchical State-Space Representation

As shown in Proposition 1, the observability of $(\mathcal{A}, \mathcal{C})$ is decoupled into those of (α_z, γ_z) and (a_I, c_L). Therefore, in the following theorem, we investigate the observability of (α_z, γ_z) because the observability of (a_I, c_L) is easily checkable. Here, define

$$\Pi := \{\pi \mid \pi \in \Lambda(a_z + \lambda b_z c_z), \lambda \in \Lambda(\Gamma_\nu)\}$$

and also define

$$\begin{aligned} \Theta_\pi &:= \{\lambda \mid \lambda \in \Lambda(\Gamma_\nu) \text{ s.t. } \pi \in \Lambda(a_z + \lambda b_z c_z)\} \\ \mathcal{V}_\pi &:= \{\xi \otimes \eta \mid \xi \in \mathfrak{E}_\lambda(\Gamma_\nu), \eta \in \mathfrak{E}_\pi(a_z + \lambda b_z c_z), \lambda \in \Theta_\pi\} \end{aligned}$$

for $\pi \in \Pi$.

Theorem 2: For $\Psi \in \mathbb{R}^{P\times N}$ and $c_G \in \mathbb{R}^{p_G\times n}$ satisfying (11), consider the system (α_z, γ_z) in (12). Then, (α_z, γ_z) is observable if and only if the following conditions hold:

(a) if $\mathrm{rank}\,(\Psi\mathsf{N}) \neq \nu$

- (i) $(\Gamma_\nu, \Psi\mathsf{N})$ is observable
- (ii) if $\mathfrak{R}(c_G^\mathsf{T}) \neq \mathfrak{R}(c_I^\mathsf{T})$, $(a_z + \lambda b_z c_z, c_G\mathsf{M})$ is observable for all $\lambda \in \Lambda(\Gamma_\nu)$
- (iii) For all $\pi \in \Pi$ such that $|\Theta_\pi| \geq 2$

$$\mathfrak{N}\,(\Psi\mathsf{N} \otimes c_G\mathsf{M}) \cap \mathfrak{R}\,(\mathcal{V}_\pi) = \{0\}. \quad (16)$$

(b) if $\mathrm{rank}\,(\Psi\mathsf{N}) = \nu$, condition (ii) holds.

Proof: [Proof of **(a)**] We can show $\Lambda(\alpha_z) = \Pi$ and $\mathcal{V}_\pi = \mathfrak{E}_\pi(\alpha_z)$ for all $\pi \in \Pi$ by the relation $\alpha_z\,(\xi \otimes \eta) = \pi\,(\xi \otimes \eta)$. Thus, (α_z, γ_z) is observable if and only if (16) holds for all $\pi \in \Pi$. By condition (iii), we need to examine this only for π such that $|\Theta_\pi| = 1$.

First, suppose $\mathfrak{R}(c_G^\mathsf{T}) = \mathfrak{R}(c_I^\mathsf{T})$. Then, there exists K such that $c_G\mathsf{M} = Kc_z$. Hence, the observability of $(a_z + \lambda b_z c_z, c_G\mathsf{M})$ is equivalent to that of (a_z, c_z). Moreover, (a_z, c_z) is observable by the definition in (13). On the other hand

$$\gamma_z\,(\xi \otimes \eta) = \Psi\mathsf{N}\xi \otimes c_G\mathsf{M}\eta \neq 0$$

holds if and only if $\Psi\mathsf{N}\xi \neq 0$ and $c_G\mathsf{M}\eta \neq 0$. From these observations, (i) and (ii) are

- necessary for (16) with arbitrary $\pi \in \Pi$,
- sufficient for (16) with all $\pi \in \Pi$ such that $|\Theta_\pi| = 1$.

This completes the proof.

[Proof of **(b)**] Let Ξ be a diagonalizing matrix such that $\Gamma_\nu \Xi = \Xi \Lambda_\nu$ with $\Lambda_\nu = \mathrm{diag}(\Lambda(\Gamma_\nu))$. Then, we have

$$\tilde{\alpha}_z = \tilde{\tilde{\Xi}}^{-1} \alpha_z \tilde{\tilde{\Xi}} = I_\nu \otimes a_z + \Lambda_\nu \otimes b_z c_z$$
$$\tilde{\gamma}_z = \gamma_z \tilde{\tilde{\Xi}} = \Psi \mathsf{N} \Xi \otimes c_G \mathsf{M}$$

where $\tilde{\tilde{\Xi}} := \Xi \otimes I_\mu$. By noting $\mathrm{rank}(\Psi \mathsf{N} \Xi) = \nu$, the observability of (α_z, γ_z) is equivalent to that of

$$(\tilde{\alpha}_z, I_\nu \otimes c_G \mathsf{M}).$$

Hence, condition (ii) is necessary and sufficient by the virtue of the block-diagonal structure. ■

This theorem shows that the observability of (α_z, γ_z) is reduced to that of the systems of smaller dimension as far as $|\Theta_\pi| \ll N$. Actually, this relation was true in all examples we tried. However, we have so far no theoretical result about the bound of $|\Theta_\pi|$. The condition (iii) is also easily checkable when the subsystems are SISO:

Corollary 1: For $\Psi \in \mathbb{R}^{P \times N}$ and $c_G \in \mathbb{R}^{1 \times n}$ satisfying (11), consider the system (α_z, γ_z) in (12) with $b_z, c_z^\mathsf{T} \in \mathbb{R}^\mu$. Then, (α_z, γ_z) is observable if and only if the following conditions hold:

(a) if $\mathrm{rank}\,(\Psi \mathsf{N}) \neq \nu$

- (i) $(\Gamma_\nu, \Psi \mathsf{N})$ is observable
- (ii) if $\Re(c_G^\mathsf{T}) \neq \Re(c_I^\mathsf{T})$, $(a_z + \lambda b_z c_z, c_G \mathsf{M})$ is observable for all $\lambda \in \Lambda(\Gamma_\nu)$
- (iii*) (a_z, b_z) is controllable.

(b) if $\mathrm{rank}\,(\Psi \mathsf{N}) = \nu$, condition (ii) holds.

Proof: [Proof of **(a)**] Conditions (i) and (ii) are identical to conditions **(a)**-(i) and (ii) in Theorem 2. Therefore, in what follows, we show the equivalence between **(a)**-(iii) in Theorem 2 and (iii*) in Corollary 1 if (a_z, b_z, c_z) is SISO.

We suppose (iii*). Note that (a_z, c_z) is observable by the definition in (13). Define polynomials $d(s) := \det(sI_\mu - a_z)$ and $n(s) := d(s) \cdot c_z (sI_\mu - a_z)^{-1} b_z$. Considering the feedback system of the scalar gain λ

$$\frac{n(s)/d(s)}{1 - \lambda \cdot n(s)/d(s)} = \frac{n(s)}{d(s) - \lambda n(s)},$$

we see that $\Lambda\,(a_z + \lambda b_z c_z)$ is identical to the set of roots of $d(s) - \lambda n(s)$. In other words, for $\lambda_1 \neq \lambda_2$,

$$\Lambda(a_z + \lambda_1 b_z c_z) \cap \Lambda(a_z + \lambda_2 b_z c_z) = \{\pi |\ d(\pi) = n(\pi) = 0\}.$$

Therefore, if $d(s)$ and $n(s)$ have no common root, or equivalently (a_z, b_z, c_z) is a minimal realization, then $|\Theta_\pi| = 1$ for all $\pi \in \Pi$. Hence, we need not examine condition **(a)**-(iii) in Theorem 2.

On the other hand, if (iii*) does not hold, there exists $\bar{\pi}$ such that $d(\bar{\pi}) = n(\bar{\pi})$. This means that $\Theta_{\bar{\pi}}$ contains all eigenvalues of Γ_ν. Therefore, $\dim(\Re(\mathcal{V}_{\bar{\pi}})) \geq \nu$. Since $\mathrm{rank}(\gamma_z) < \nu$, (16) cannot hold. This completes the proof.

[Proof of **(b)**] From the same argument as in the proof of **(b)** in Theorem 2, the result follows. ■

Corollary 1, which is a generalized result derived in [15] where the case of $c_G = c_I$ is considered, gives a simple condition when the subsystem is SISO and $c_G \in \mathbb{R}^{1 \times n}$. Furthermore, from the point of view of determining the output matrices such as Ψ and c_G, these statements are useful. For example, if only condition (i) is not satisfied, we only have to replace Ψ to make *overall* network system observable. In addition, Corollary 1 shows that if (a_z, b_z) is uncontrollable, then (α_z, γ_z) is necessarily unobservable in this specific case, independently of the choice of the output matrices Ψ and c_G.

III. Numerical Example

In this section, we validate the proposed **HD**-observer through a simple numerical example. Here, the state matrix A in (3) is supposed to be given by the subsystem (a_I, b_I, c_I) and the network structure Γ as

$$a_I = \begin{bmatrix} -3.8 & 1 & 1 & 1 & 1 \\ 0 & -1.8 & 2 & 0 & 0 \\ 0 & 0 & -1.8 & 2 & 0 \\ 0 & 0 & 0 & -1.8 & 2 \\ 0 & 2 & 0 & 0 & -1.8 \end{bmatrix}$$
$$b_I = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^\mathsf{T},\ c_I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\Gamma = \begin{bmatrix} -2 & 2 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Furthermore, the output matrix C in (4) is given by $c_G = c_I$ and

$$\Psi = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix},\ c_L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

with $\mathcal{I} = \{4\}$. This means that the global output y^G is measured from the first and second subsystems with c_G, and the local output $y^L_\mathcal{I}$ is measured from the fourth subsystem with c_L, whose internal state is to be estimated. The global structure of the network system (A, C) is depicted in Fig. 4. Here, the rank of the observability matrix $\mathcal{O}_{20}(A, C)$ is 11. It should be emphasized that we cannot determine whether the state on the fourth subsystem is estimated with usual *centralized* observers.

We consider deriving the **HSS**-representation $(\mathcal{A}, \mathcal{C})$ in (5) and (12) by Theorem 1. Here, the rank of Γ is $\nu = 3$, and the dimension of the observable sub-space of $(a_I, b_I c_I)$ is $\mu = 2$. Therefore, the dimension of z is $\nu\mu = 6$. Then, by using the projection of

$$\mathsf{N} = \begin{bmatrix} -0.71 & -0.41 & -0.29 \\ 0.71 & 0.41 & -0.29 \\ 0 & 0.82 & -0.29 \\ 0 & 0 & 0.87 \end{bmatrix},\ \mathsf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \\ 0 & 0.5 \\ 0 & 0.5 \end{bmatrix}$$

satisfying (10), we have

$$\Gamma_\nu = \begin{bmatrix} -4 & 1.73 & 0 \\ 0.58 & -3.33 & 1.89 \\ -0.20 & 0.83 & -1.67 \end{bmatrix}$$

and

$$a_z = \begin{bmatrix} -3.8 & 2 \\ 0 & 0.2 \end{bmatrix},\ b_z = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix},\ c_z = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

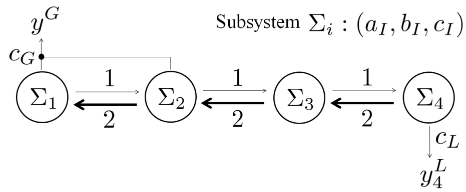

Fig. 4. Global structure of network system (A, C).

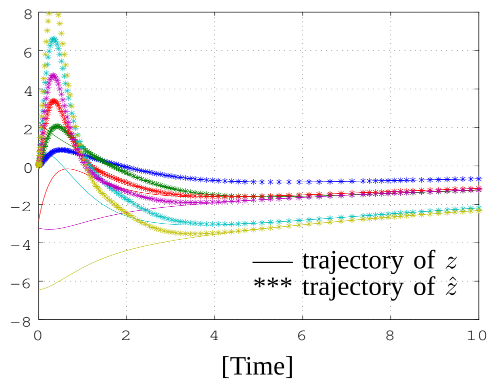

Fig. 5. Trajectory of states z and $\hat{z}$.

Clearly, Ψ and c_G satisfy (11) from $c_G = c_I$, and Ψ is linearly dependent on the first row of Γ.

Next, we examine the observability of $(\mathcal{A}, \mathcal{C})$, which is decoupled into the the observability of (a_I, c_L) and (α_z, γ_z) in (12). The observability of (α_z, γ_z) is verified by Corollary 1. In fact, both (a_I, c_L) and (α_z, γ_z) are observable. Therefore, there exists some feedback gains H and h_4 in (8) such that any convergence rate for the estimation of $z \in \mathbb{R}^6$ and $x_4 \in \mathbb{R}^5$ is achieved.

We design some feedback gains H and h_4 for the **HD**-observer in (8) by the pole placement via MATLAB. Here, Fig. 5 shows the trajectory of $z \in \mathbb{R}^6$ (the solid lines) and the estimation $\hat{z} \in \mathbb{R}^6$ (the lines of $*$), and Fig. 6 shows the trajectory of $x \in \mathbb{R}^{20}$ (the solid lines) and the estimation $\hat{x}_4 \in \mathbb{R}^5$ (the lines of $*$) under some initial values. From these figures, we can see that z, which represents the interaction of the subsystems, is appropriately estimated, and the decentralized estimation of the states on Σ_4 is achieved.

IV. Conclusion

In this paper, the problem of a hierarchical decentralized observer design for networked systems in which identical subsystems are linearly coupled under a network, has been addressed. First, to design the observer, we have derived a hierarchical model composed of the dynamics on interaction among the subsystems and the dynamics on subsystems whose states are to be estimated. Based on the hierarchical model, the hierarchical decentralized observer, where an observer for the interaction and an observer for each subsystem are hierarchically interconnected, has been designed. In addition, the observability condition for realizing the hierarchical decentralized observer has been derived.

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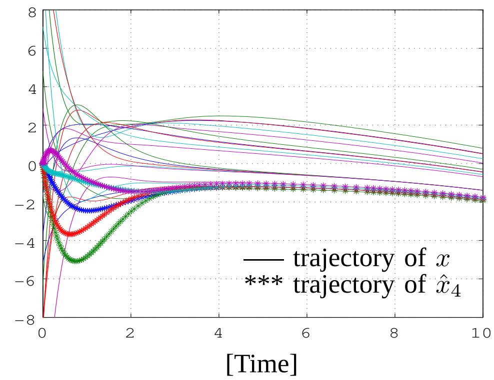

Fig. 6. Trajectory of states x_1 and $\hat{x}_4$.

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