Model Order Reduction for MIMO Linear Dynamical Networks via Reaction-Diffusion Transformation

Takayuki Ishizaki, Kenji Kashima, Jun-ichi Imura* and Kazuyuki Aihara**

***Abstract*— In this paper, we propose a model order reduction method for MIMO linear dynamical networks, where a large number of subsystems interact according to a network. In this method, a spatially one-dimensional reaction-diffusion structure, which can be efficiently exploited even for large-scale systems, is fully utilized. We give a simple algorithm as well as a computable error bound in terms of the H_∞-norm.**

I. Introduction

Dynamical systems on large-scale/complex networks, whose behaviors are determined by the interaction of a large number of subsystems, have been intensively studied over the past decades. Examples of such dynamical networks include World Wide Web, gene regulatory networks, spread of infection; see [1], [2], [3] for an overview.

This paper addresses the model reduction problem of MIMO linear systems over large-scale networks expressed by undirected graphs, whose nodes and edges denote subsystems and their interactions, respectively. As one of approach to solve the reduction problem, we exploit a network structure transformation that we call the Reaction-Diffusion transformation. This transformation gives a kind of spatially *one-dimensional* reaction-diffusion structure embedded in the networks. To execute the Reaction-Diffusion transformation, we use Householder transformation, which is effective for making large-scale (symmetric) matrices to band matrices. Thus, the proposed model reduction method is effective for large-scale linear systems on undirected graphs.

Many kinds of model reduction methods of linear and nonlinear systems have been developed [4], [5], [6], [7], [8]. However, matrix factorization methods such as the balanced truncation require computationally expensive operations, e.g., gramian computations, although the stability/passivity property is preserved [4], [5]. Therefore, these methods are in general difficult to apply to large-scale systems. On the other hand, Krylov method is also well-known as a model reduction method for large-scale systems. However, in this method, the stability property is not preserved, and a priori computable error bound caused via model reduction has not been derived even in linear systems [5], [6].

Most recently, we have proposed in [9] a new model reduction method for SISO linear dynamical network systems by focusing on the reaction-diffusion structure. Even though the method can be applied only for SISO systems, it has the stability preservation and provides an explicit error bound caused via the model reduction, which can be computed even for large-scale systems by efficient iterative matrix computations.

The main results of this paper are generalization of the model reduction strategies for SISO systems proposed in [9] to MIMO systems. It should be emphasized that the quantitative analysis of the reduction error for MIMO systems is even more difficult. Specifically, the analytical results of the H_∞-norm for SISO systems were derived in [9] only by utilizing elementary properties of rational transfer functions. In contrast, we derive in this paper more general expression of the H_∞-norm condition for MIMO systems by exploiting the bounded real lemma [5].

This paper is organized as follows. In section II, we describe a system to be studied here and introduce the Reaction-Diffusion transformation for MIMO systems. In section III, we analyze the properties of the Reaction-Diffusion realization and apply the properties to a model order reduction. In Section IV, a numerical example validates the proposed method. Section V concludes this paper.

NOTATION: For a vector v and a matrix $M = \{m_{ij}\}$, we use the following notation in this paper:

$\mathbb{R}$	the set of real numbers
I_n	the unit matrix of the size $n \times n$
e_k^n	the k-th column vector of I_n
$e_{k_1:k_2}^n$	the k_1-th to k_2-th columns of I_n
$M \otimes N$	the Kronecker product of M and N, namely $\{m_{ij}N\}$
$\|M\| = \sigma_{\max}(M)$	the maximum singular value of M
$\det(M)$	the determinant of M
$\mathrm{abs}(M)$	the matrix formed by $\{\|m_{ij}\|\}$

In addition, the product of matrices is defined by $\prod_{k=i}^{j} M_k := M_i M_{i+1} \cdots M_j$ and if $j < i$ this is equal to 0. Finally, the H_∞-norm of a rational transfer matrix $G(s)$ is defined by $\|G(s)\|_\infty := \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(j\omega))$.

II. Reaction-Diffusion Transformation

A. System Description

In this paper, we deal with linear systems over large-scale complex networks whose general form is given as follows:

Definition 1: The linear system

$$\dot{x} = Ax + Bu, \quad x(0) = 0 \tag{1}$$

with $A = \{a_{i,j}\} \in \mathbb{R}^{N \times N}$ and $B = \{b_{i,j}\} \in \mathbb{R}^{N \times m}$ is said to be a *dynamical network* (A, B) if A is stable and symmetric, and $N = nm$ for some integer n.

*Graduate School of Information Science and Engineering, Tokyo Institute of Technology; 2-12-1, Meguro ward, Tokyo {ishizaki,kashima,imura}@cyb.mei.titech.ac.jp

**Institute of Industrial Science, University of Tokyo; 4-6-1 Komaba, Meguro ward, Tokyo aihara@sat.t.u-tokyo.ac.jp

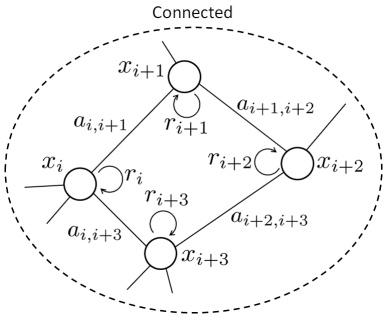

Fig. 1. Illustration of network structure

This is a generalization of *undirected reaction-diffusion* systems depicted in Fig. 1:

$$\dot{x}_i = -r_i x_i + \sum_{j=1, j\neq i}^{N} a_{i,j}\left(x_j - x_i\right) + \sum_{k=1}^{m} b_{i,k} u_k \quad (2)$$

where $r_i(\geq 0)$ denotes the intensity of the reaction (chemical dissolution) of x_i, and $a_{i,j}(\geq 0)$, $i \neq j$, denotes the intensity of the diffusion between x_i and x_j. This coupled dynamics is stable if at least one r_i is strictly positive and the graph is connected. See, e.g., [3] for a survey on networked systems and multi-agent systems.

Next, we introduce a class of the state-space realization of the dynamical network. This representation plays a central role in this paper and is closely related to spatially one-dimensional reaction-diffusion systems.

Definition 2: Let (A, B) a dynamical network in (1). Then, unitary H is said to be a *Reaction-Diffusion transformation matrix* if $\mathcal{A} := HAH^{\mathsf{T}}$ and $\mathcal{B} := HB$ are in the form of

$$\begin{aligned}
\mathcal{A} &= \begin{bmatrix} \alpha_1 & \beta_1^{\mathsf{T}} & & & \\ \beta_1 & \alpha_2 & \beta_2^{\mathsf{T}} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{n-1}^{\mathsf{T}} \\ & & & \beta_{n-1} & \alpha_n \end{bmatrix} \in \mathbb{R}^{N\times N} \\
\mathcal{B} &= \begin{bmatrix} \beta_0^{\mathsf{T}} & 0 & \cdots & 0 \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{N\times m}
\end{aligned} \quad (3)$$

with some negative definite matrices $\alpha_i \in \mathbb{R}^{m\times m}$ for $i \in \{1, \cdots, n\}$ and some upper triangular matrices with non-negative diagonal entries $\beta_i \in \mathbb{R}^{m\times m}$ for $i \in \{0, \cdots, n-1\}$. Moreover, the realization $(\mathcal{A}, \mathcal{B})$ is called a *Reaction-Diffusion realization* [1].

Note that the matrix $\mathcal{A}$ is a band matrix with the bandwidth of $2m+1$ since β_i is an upper triangular and $\mathcal{A}$ has a block tri-diagonal structure with $m \times m$ blocks. For the **RD**-realization, we denote the state vector $\mathcal{X} := Hx \in \mathbb{R}^N$ as $\mathcal{X} := [\mathcal{X}_1^{\mathsf{T}}, \cdots, \mathcal{X}_n^{\mathsf{T}}]^{\mathsf{T}}$, where $\mathcal{X}_i \in \mathbb{R}^m$. The internal structure represents serially-cascaded autonomous systems equipped with the boundary input, as shown in Fig. 2. In fact, the **RD**-realization appears when we apply the finite difference method to (multi-variable) spatially one-dimensional reaction-diffusion systems with a boundary input.

[1]The term "Reaction-Diffusion" is , as necessary, denoted as "**RD**-".

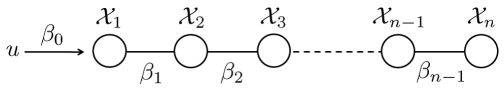

Fig. 2. Illustration of **RD**-realization

Theorem 1: Consider the dynamical network (A, B) in (1). There exists a Householder matrix H_B that makes $H_B B$ an upper triangular matrix with the non-negative diagonal entries. Furthermore, there exists a Householder matrix H_A that makes $H_A H_B A H_B H_A$ a band matrix with the bandwidth of $(2m+1)$ and the non-negative $(i, i+m), (i+m, i)$-th entries for all $i \in \{1, \cdots, (n-1)m\}$. Then, the unitary matrix

$$H = H_A H_B \quad (4)$$

is a **RD**-realization matrix of the dynamical network (A, B). Moreover, define

$$\bar{i} := \begin{cases} \min_i\{i : \beta_i = 0\} \text{ if } \prod_{i=1}^{n-1} \beta_i = 0, \\ n, \text{ otherwise.} \end{cases} \quad (5)$$

Then, $(e^{nm}_{1:m\bar{i}})^{\mathsf{T}} H e^{nm}_{1:m\bar{i}}$ is uniquely determined.

Proof: See Chapter 5 in [10] for the details of the Householder transformation. The proof of the **RD**-realization is similar to that of Theorem 1 in [9]. Furthermore, any non-singular matrix can be decomposed to the product of an orthogonal matrix and an upper triangular matrix with positive diagonal terms, namely QR decomposition. By noting the uniqueness of the QR decomposition of B_i, the uniqueness of H follows form similar argument as in [9]. (The details are omitted for space.) ■

In computer science, matrices are often transformed to an easily-handled form via similar transformations. In particular for symmetric matrices, unitary transformations are desirable to retain the symmetry. It should be emphasized that the construction of the Householder matrices does not require computationally expensive operations, such as matrix factorizations [2]. Moreover, construction methods of the band matrix for large matrices have been widely investigated in the computer science since Householder transformations has various application such as eigenvalue computation [11]. In this sense, the **RD**-transformation can be implemented even for large-scale systems.

B. Fundamental Properties of MIMO Reaction-Diffusion Realization

In this section, we show a *low-pass* property equipped with the **RD**-realization. In the rest of this paper, the transfer function matrix from the input $u \in \mathbb{R}^m$ to the state $\mathcal{X}_i \in \mathbb{R}^m$ in the **RD**-realization is denoted by

$$\mathcal{G}_i(s) := \left(e_i^n \otimes I_m\right)^{\mathsf{T}} \left(sI_{nm} - \mathcal{A}\right)^{-1} \mathcal{B}. \quad (6)$$

[2]In fact, the Householder transformations require only $(2/3)n^3$ computations for the size of matrices n while, e.g., the computation of the controllability gramians and the eigen-decomposition require $70n^3$ and $30n^3$ computations. See [11], [5] for details.

Theorem 2: Let $(\mathcal{A},\mathcal{B})$ be a **RD**-realization in (3). Then, $\mathcal{G}_i(s)$ in (6) satisfies

$$\mathcal{G}_i(s)=\prod_{j=1}^{i}\mathcal{P}_j(s)\beta_{j-1}, \tag{7}$$

$$\mathcal{P}_j(s):=\left(e_1^{p_{n,j}}\otimes I_m\right)^{\mathsf{T}}\left(sI_{p_{n,j}m}-\overline{\mathcal{A}}_j\right)^{-1}\left(e_1^{p_{n,j}}\otimes I_m\right)$$

where $p_{n,j}:=n-j+1$ and $\overline{\mathcal{A}}_j:=(e_{j:n}^n\otimes I_m)^{\mathsf{T}}\mathcal{A}(e_{j:n}^n\otimes I_m)$. Moreover

$$\|\mathcal{P}_i(s)\|_\infty=\|\mathcal{P}_i(0)\|,\quad \forall i\in\{1,\cdots,n\} \tag{8}$$

holds.

Proof: [Proof of (7)] Trivially from the definitions of $\mathcal{G}_i$ and $\mathcal{P}_i$, we have $\mathcal{G}_1=\mathcal{P}_1\beta_0$, which implies

$$\mathcal{X}_1=\mathcal{P}_1(s)\beta_0 u.$$

Suppose $\mathcal{X}_i=\prod_{j=1}^{i}\mathcal{P}_j\beta_{j-1}u$. By regarding $\mathcal{X}_i$ as an input to the state equation of $\mathcal{X}_{i+1}$, we have

$$\mathcal{X}_{i+1}=\mathcal{P}_{i+1}(s)\beta_i\left(\prod_{j=1}^{i}\mathcal{P}_j(s)\beta_{j-1}u\right).$$

Hence, (7) holds by the induction. Lemma 1 below includes (8) as a special case. ∎

Lemma 1: For any stable $A=A^{\mathsf{T}}\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times m}$ with $m\le n$

$$\left\|B^{\mathsf{T}}(sI_n-A)^{-1}B\right\|_\infty=\left\|B^{\mathsf{T}}A^{-1}B\right\| \tag{9}$$

holds.

Proof: Denote as $f(s):=B^{\mathsf{T}}(sI_n-A)^{-1}B$. The condition $\|f(s)\|_\infty\le\gamma$ holds true if and only if the Hamiltonian

$$J(\gamma)=\begin{bmatrix}A & \frac{1}{\gamma}BB^{\mathsf{T}}\\ -\frac{1}{\gamma}BB^{\mathsf{T}} & -A\end{bmatrix}$$

has no eigenvalues on the imaginary axis (see [5] for the details). Here, we have

$$\det(J(\gamma))=-\det(A)^2\det\left(I_n-\frac{1}{\gamma^2}f(0)^2\right).$$

This implies that $J(\gamma)$ has necessarily zero eigenvalues if $\gamma=\|f(0)\|$. Therefore, due to the continuity of $J(\gamma)$, the result follows if the eigenvalues of $J(\gamma)$ are real or purely imaginary. For the symmetric positive definite matrix $S:=(1/\gamma)BB^{\mathsf{T}}$ we have

$$J^2(\gamma)=I_2\otimes\left(A^2-S^2\right)+\overline{I}_2\otimes(AS-SA),\ \overline{I}_2=\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

Let a matrix $V\in\mathbb{R}^{n\times n}$ such that $\overline{I}_2V=VI_2$. By noting that $A-S=(A-S)^{\frac{1}{2}}(A-S)^{\frac{1}{2}}$ is negative definite, the similar transformation of $J^2(\gamma)$ by $V\otimes(A-S)^{\frac{1}{2}}$ yields

$$\begin{aligned}&\left\{V^{-1}\otimes(A-S)^{-\frac{1}{2}}\right\}J^2(\gamma)\left\{V\otimes(A-S)^{\frac{1}{2}}\right\}\\&=I_2\otimes\left\{(A-S)^{-\frac{1}{2}}\left(A^2-S^2+AS-SA\right)(A-S)^{\frac{1}{2}}\right\}\\&=I_2\otimes\left\{(A-S)^{\frac{1}{2}}(A+S)(A-S)^{\frac{1}{2}}\right\}.\end{aligned}$$

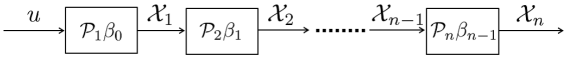

Fig. 3. Block diagram of **RD**-realization

Since $(A-S)^{1/2}(A+S)(A-S)^{1/2}$ is the Hermitian matrix, having real eigenvalues, the result follows. ∎

Fig. 3 depicts the structure of the **RD**-realization shown in Theorem 2.

III. Application to Model Order Reduction

A. Approximation Error Analysis

In this section, we propose a model reduction method based on the direct truncation of the **RD**-realization. We simply retain the upper km states, and analyze a resulting error bound.

Theorem 3: Consider the transfer function matrix $\mathcal{G}_i(s)$ in (6) and define the km-th order transfer function matrix

$$\hat{\mathcal{G}}_i^{(k)}(s):=\left(e_i^k\otimes I_m\right)^{\mathsf{T}}(sI_{km}-\mathcal{A}_k)^{-1}\mathcal{B}_k \tag{10}$$

where

$$\mathcal{A}_k:=(e_{1:km}^{nm})^{\mathsf{T}}\mathcal{A}e_{1:km}^{nm},\quad \mathcal{B}_k:=(e_{1:km}^{nm})^{\mathsf{T}}\mathcal{B}. \tag{11}$$

Then, for all $i\in\{1,\cdots,n\}$ and $k\in\{1,\cdots,n-1\}$

$$\begin{aligned}&\left\|\mathcal{G}_i(s)-\hat{\mathcal{G}}_i^{(k)}(s)\right\|_\infty \\&\le\begin{cases}\rho_{k+1}\sum_{j=1}^{i}\left\{\phi_j\left(\prod_{l=j}^{k}\rho_l^2\phi_l^2\right)\left(\prod_{l=1,l\ne j}^{i}\rho_l\phi_{l-1}\right)\right\}, & i\le k\\ \prod_{j=1}^{i}\rho_j\phi_{j-1}, & i\ge k+1\end{cases}\end{aligned} \tag{12}$$

holds, where

$$\rho_i:=\|\mathcal{P}_i(0)\|,\quad \phi_i:=\|\beta_i\|. \tag{13}$$

In particular, if $m=1$

$$\left\|\mathcal{G}_i(s)-\hat{\mathcal{G}}_i^{(k)}(s)\right\|_\infty=\mathcal{G}_i(0)-\hat{\mathcal{G}}_i^{(k)}(0) \tag{14}$$

holds.

Proof: For an integer $k\in\{1,\cdots,n-1\}$, denote by $\mathcal{G}_i(s;\tau_k)$ the transfer function $\mathcal{G}_i(s)$ for which β_k is replaced with $\tau_k\beta_k$ for $\tau_k\in[0,1]$. Here, note that $\mathcal{G}_i(s;\tau_k=1)=\mathcal{G}_i(s)$ and $\mathcal{G}_i(s;\tau_k=0)=\hat{\mathcal{G}}_i^{(k)}(s)$. For $i\ge k+1$, $\hat{\mathcal{G}}_i^{(k)}(s)=0$ and consequently the desired result holds from Theorem 2. Thus, in what follows, we assume $i\le k$.

Similarly to $\mathcal{G}_i(s;\tau_k)$, denote by $\mathcal{P}_i(s;\tau_k)$ the function $\mathcal{P}_i(s)$ for which β_k is replaced with $\tau_k\beta_k$ for $\tau_k\in[0,1]$. From (7), using the differential formula for a non-singular matrix, we have

$$\frac{d\mathcal{P}_i(s;\tau_k)}{d\tau_k}=2\tau_k\left(\prod_{j=i}^{k}\mathcal{Q}_j(s;\tau_k)\right)^{\mathsf{T}}\mathcal{P}_{k+1}(s;\tau_k)\prod_{j=i}^{k}\mathcal{Q}_j(s;\tau_k) \tag{15}$$

where $\mathcal{Q}_j(s;\tau_k):=\beta_j\mathcal{P}_j(s;\tau_k)$. Furthermore, from the semi-negative definiteness of $\mathcal{P}_i(0;\tau_k)$, we have

$$\|\mathcal{P}_i(0;\tau_k=1)\|\ge\|\mathcal{P}_i(0;\tau_k)\|,\quad\forall\tau_k\in[0,1]. \tag{16}$$

First, we consider the case of $m = 1$. Noting all the transfer functions and β_i are scalars, we assume

$$\left|\frac{d\mathcal{G}_i(s;\tau_k)}{d\tau_k}\right| \le \frac{d\mathcal{G}_i(0;\tau_k)}{d\tau_k} \tag{17}$$

then we have

$$\left\|\mathcal{G}_i(s) - \hat{\mathcal{G}}_i^{(k)}(s)\right\|_\infty = \left|\int_0^1 \frac{d\mathcal{G}_i(s;\tau_k)}{d\tau_k} d\tau_k\right|$$
$$\le \int_0^1 \frac{d\mathcal{G}_i(0;\tau_k)}{d\tau_k} d\tau_k = \mathcal{G}_i(0;\tau_k=1) - \mathcal{G}_i(0;\tau_k=0).$$

Therefore, what remains to be shown is (17). From the differentiation of $\mathcal{G}_i(s;\tau_k) = \prod_{j=1}^i \mathcal{P}_j(s;\tau_k)\beta_{j-1}$ and (15), we have

$$\frac{d\mathcal{G}_i(s;\tau_k)}{d\tau_k} = \sum_{j=1}^i \left\{ \frac{d\mathcal{P}_j(s)}{d\tau_k}\beta_{j-1} \prod_{l=1,l\neq j}^i \mathcal{P}_l(s)\beta_{l-1} \right\}$$
$$= 2\tau_k \mathcal{P}_{k+1}(s) \sum_{j=1}^i \left\{ \beta_{j-1}\left(\prod_{l=j}^k \mathcal{P}_l(s)^2\beta_l^2\right)\left(\prod_{l=1,l\neq j}^i \mathcal{P}_l(s)\beta_{l-1}\right)\right\}$$
$$=: 2\tau_k f_{i,k}^{(1)}(\beta_0,\cdots,\beta_k,\mathcal{P}_1(s;\tau_k),\cdots,\mathcal{P}_{k+1}(s;\tau_k)).$$

Noting the function $f_{i,k}^{(1)}$ is given by the summation and multiplication of β_i and $\mathcal{P}_i(s;\tau_k)$, where $\mathcal{P}_i(0;\tau_k) \ge 0$ and $\beta_i \ge 0$, we have from (8)

$$\left|\frac{d\mathcal{G}_i(s;\tau_k)}{d\tau_k}\right| = \left|2\tau_k f_{i,k}^{(1)}(\beta_0,\cdots,\mathcal{P}_{k+1}(s;\tau_k))\right|$$
$$\le 2\tau_k f_{i,k}^{(1)}(\beta_0,\cdots,\mathcal{P}_{k+1}(0;\tau_k)) = \frac{d\mathcal{G}_i(0;\tau_k)}{d\tau_k}.$$

Then, (17) follows.

Next, we consider the case of $m \ge 2$. We have

$$\left\|\mathcal{G}_i(s) - \hat{\mathcal{G}}_i^{(k)}(s)\right\|_\infty \le \int_0^1 \left\|\frac{d\mathcal{G}_i(s;\tau_k)}{d\tau_k}\right\|_\infty d\tau_k. \tag{18}$$

We can see from

$$\frac{d\mathcal{G}_i(s;\tau_k)}{d\tau_k}$$
$$= \sum_{j=1}^i \left\{ \left(\prod_{l=j+1}^i \mathcal{P}_l(s)\beta_{l-1}\right)\frac{d\mathcal{P}_j(s)}{d\tau_k}\beta_{j-1}\left(\prod_{l=1}^{j-1}\mathcal{P}_l(s)\beta_{l-1}\right)\right\}$$
$$=: 2\tau_k f_{i,k}^{(m)}(\beta_0,\cdots,\beta_k,\mathcal{P}_1(s;\tau_k),\cdots,\mathcal{P}_{k+1}(s;\tau_k))$$

and (15) that the function $f_{i,k}^{(m)}$ is given by the summation and the multiplication of β_i and $\mathcal{P}_i(s;\tau_k)$. In addition, since the function $f_{i,k}^{(m)}$ can be interpreted as an extended function of $f_{i,k}^{(1)}$, whose domain is extended to $m\times m$ matrix, (8) and (16) yield

$$\left\|\frac{d\mathcal{G}_i(s;\tau_k)}{d\tau_k}\right\|_\infty = \left\|2\tau_k f_{i,k}^{(m)}(\beta_0,\cdots,\mathcal{P}_{k+1}(s;\tau_k))\right\|_\infty$$
$$\le 2\tau_k f_{i,k}^{(1)}(\|\beta_0\|,\cdots,\|\mathcal{P}_{k+1}(s;\tau_k)\|_\infty)$$
$$= 2\tau_k f_{i,k}^{(1)}(\phi_0,\cdots,\phi_k,\rho_1,\cdots,\rho_{k+1}).$$

By substituting this inequality into (18) and the integration, the result follows. ■

Theorem 3 states that for single-input systems, we have the exact error. Here, (12) seems inconvenient for the calculation due to its complicate form. The following corollary gives an equivalent expression of the error bound (12) as a closed matrix form:

Corollary 1: For the constants

$$\theta_n := -\frac{1}{\rho_n},\quad \theta_i := -\frac{1}{\rho_i} - \phi_i^2\rho_{i+1},\ i\in\{1,\cdots,n-1\}$$

with ϕ_i and ρ_i in (13), define

$$\Theta_{\langle k\rangle}(\tau) := \begin{bmatrix} \theta_1 & \phi_1 & & & & & \\ \phi_1 & \theta_2 & \ddots & & & & \\ & \ddots & \ddots & \tau\phi_k & & & \\ & & \tau\phi_k & \ddots & \ddots & & \\ & & & \ddots & \ddots & \phi_{n-1} & \\ & & & & \phi_{n-1} & \theta_n \end{bmatrix} \in \mathbb{R}^{n\times n}$$
$$\Phi := \begin{bmatrix} \phi_0 & 0 & \cdots & 0 \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{n\times 1} \tag{19}$$

and $\Theta := \Theta_{\langle 1\rangle}(1) = \cdots = \Theta_{\langle n-1\rangle}(1)$. Then [3]

$$\left\|\mathcal{G}_i(s) - \hat{\mathcal{G}}_i^{(k)}(s)\right\|_\infty \le \begin{cases} -\frac{1}{2}(e_i^n)^{\mathsf{T}} \left.\frac{d\Theta_{\langle k\rangle}^{-1}(\tau)}{d\tau}\right|_{\tau=1}\Phi, & i\le k \\ -(e_i^n)^{\mathsf{T}}\Theta^{-1}\Phi, & i\ge k+1 \end{cases} \tag{20}$$

is equivalent to (12).

Proof: For all $i \le k$, we can see from the proof of Theorem 3 that

$$\left\|\mathcal{G}_i(s) - \hat{\mathcal{G}}_i^{(k)}(s)\right\|_\infty \le f_{i,k}^{(1)}(\phi_0,\cdots,\phi_k,\rho_1,\cdots,\rho_{k+1}).$$

Note that $\left(\Theta_{\langle k\rangle}(\tau),\Phi\right)$ is a **RD**-realization for $m=1$. When $\tau = 1$, considering the recurrence formula in (7) associated with $\left(\Theta_{\langle k\rangle}(\tau),\Phi\right)$, we have $\mathcal{P}_n(0) = -1/\theta_n = \rho_n$. Here, the supposition of $P_i(0) = \rho_i$ yields inductively

$$\mathcal{P}_{i-1}(0) = \frac{1}{-\theta_{i-1} - \phi_{i-1}^2\mathcal{P}_i(0)} = \rho_{i-1},$$

and therefore $\mathcal{P}_i(0) = \mathcal{P}_i(0;\tau=1) = \rho_i$ holds for all $i\in\{1,\cdots,n\}$. By substituting $\tau = 1$ into

$$\frac{d}{d\tau}\left(-(e_i^n)^{\mathsf{T}}\Theta_{\langle k\rangle}^{-1}(\tau)\Phi\right) = 2\tau f_{i,k}^{(1)}(\phi_0,\cdots,\mathcal{P}_{k+1}(0;\tau)),$$

the result follows. Next, for all $i \le k+1$ we have

$$-(e_i^n)^{\mathsf{T}}\Theta^{-1}\Phi = \prod_{j=1}^i \mathcal{P}_j(0)\phi_{j-1} = \prod_{j=1}^i \rho_j\phi_{j-1}.$$

Therefore, the result follows. ■

Remark 1: We can approximately compute the differential coefficient in (20) by

$$\left.\frac{d\Theta_{\langle k\rangle}^{-1}(\tau)}{d\tau}\right|_{\tau=1}\Psi \simeq \frac{\Theta^{-1} - \Theta_{\langle k\rangle}^{-1}(1-\Delta\tau)}{\Delta\tau}\Psi \tag{21}$$

[3] $\Theta_{\langle k\rangle}^{-1}(\tau)$ denotes $\{\Theta_{\langle k\rangle}(\tau)\}^{-1}$

for some small $\Delta\tau$. It should be emphasized that the value of (21) can be efficiently found by exploiting the structure of $\Psi,\ \Theta$ and $\Theta_{\langle k\rangle}\left(1-\Delta\tau\right)$ [5].

In the above arguments, we discuss the model reduction of the mapping form the input to the states. These results immediately lead the result for the input-to-output mapping as shown in the following corollary:

Corollary 2: Consider the **RD**-realization $(\mathcal{A},\mathcal{B})$ of the dynamical network (1) with the output mapping $y=\mathcal{C}\mathcal{X}$, where $\mathcal{C}=[c_1,\cdots,c_n]\in\mathbb{R}^{1\times nm}$ for $c_i\in\mathbb{R}^{1\times m}$. Denote the transfer function matrix and its km-th order model by

$$\begin{aligned} G\left(s\right) &:= \mathcal{C}\left(sI_{nm}-\mathcal{A}\right)^{-1}\mathcal{B} \\ \hat{G}^{(k)}\left(s\right) &:= \mathcal{C}_k\left(sI_{km}-\mathcal{A}_k\right)^{-1}\mathcal{B}_k \end{aligned}$$

where $\mathcal{A}_k$ and $\mathcal{B}_k$ in (11), and $\mathcal{C}_k:=\mathcal{C}e^{nm}_{1:km}$. Then, $\hat{G}^{(k)}\left(s\right)$ is stable and

$$\left\|G\left(s\right)-\hat{G}^{(k)}\left(s\right)\right\|_\infty \leq \Psi\Xi_k \tag{22}$$

holds, where

$$\Xi_k := \left[\begin{array}{c}\xi_k\\ \eta_{n-k}\end{array}\right],\quad \left\{\begin{array}{l}\xi_k=-\dfrac{1}{2}\left(e^n_{1:k}\right)^{\mathsf{T}}\left.\dfrac{d\Theta^{-1}_{\langle k\rangle}\left(\tau\right)}{d\tau}\right|_{\tau=1}\Phi\\ \eta_{n-k}=-\left(e^n_{k+1:n}\right)^{\mathsf{T}}\Theta^{-1}\Phi\end{array}\right.$$

$$\Psi := \left[\begin{array}{ccc}\|c_1\| & \cdots & \|c_n\|\end{array}\right].$$

Moreover, if $m=1$

$$\left\|G\left(s\right)-\hat{G}^{(k)}\left(s\right)\right\|_\infty \leq \left|\mathrm{abs}\left(\mathcal{C}_k\right)\mathcal{A}_k^{-1}\mathcal{B}_k-\mathrm{abs}\left(\mathcal{C}\right)\mathcal{A}^{-1}\mathcal{B}\right| \tag{23}$$

holds. In particular, if the elements of $\mathcal{C}$ have the same sign, the relation (23) holds with the equality.

Proof: The proof of the stability of $\hat{G}^{(k)}\left(s\right)$ is trivial thanks to the negative definiteness of $\mathcal{A}$. Noting $\hat{\mathcal{G}}^{(k)}_i\left(s\right)=0$ for all $i\geq k+1$, we obtain

$$\left\|G\left(s\right)-\hat{G}^{(k)}\left(s\right)\right\|_\infty \leq \sum_{j=1}^{n}\|c_j\|\left\|\mathcal{G}_j\left(s\right)-\hat{\mathcal{G}}^{(k)}_j\left(s\right)\right\|_\infty.$$

Therefore, (22) follows from Corollary 1. (See [9] for the case of $m=1$.) ■

Corollary 2 gives the upper bound of the approximation error of the input-to-output mapping. For the model reduction of the dynamical network (1) with the output mapping $y=Cx$, we calculate $\mathcal{C}$ as $\mathcal{C}=CH^{\mathsf{T}}$. Furthermore, for the multi-output systems, we can simply evaluate (22) or (23) for all outputs.

Remark 2: The proposed model reduction method is mathematically similar to the model reduction based on Arnoldi algorithm in Krylov method (see, e.g., [5], [13]). However, Krylov method in general does not produce a band matrix in which diagonal entries in off-diagonal block matrices have the same sign. Moreover, the Krylov methods cannot provide an a priori error bound for the reduction. On the contrary, the proposed method uses Householder transformation to make the diagonal entries in the off-diagonal block matrices non-negative. The non-negativity enables to derive the error bound in (22).

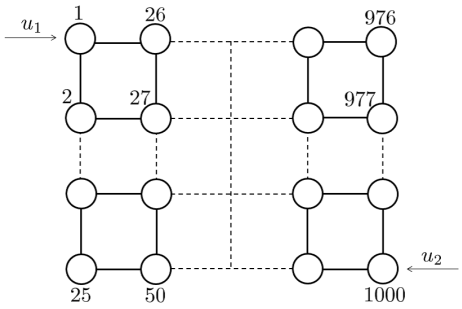

Fig. 4. Dynamical network on two-dimensional lattice; 25×40 nodes

The algorithm of the proposed model order reduction method is as follows:

(a) Transform the dynamical network (1) with the outputs $y=Cx$ to the **RD**-realization $(\mathcal{A},\mathcal{B})$ with $\mathcal{C}=CH^{\mathsf{T}}$ by Theorem 1.

(b) Give a positive constant ε as the upper bound of the approximation error.

(c) By Corollary 2, find the minimum k satisfying for all outputs

$$\begin{aligned} &\frac{\mathrm{abs}\left(\mathcal{C}_k\right)\mathcal{A}_k^{-1}\mathcal{B}_k-\mathrm{abs}\left(\mathcal{C}\right)\mathcal{A}^{-1}\mathcal{B}}{-\mathrm{abs}\left(\mathcal{C}\right)\mathcal{A}^{-1}\mathcal{B}}<\varepsilon,\quad \text{if } m=1\\ &\frac{\Psi\Xi_k}{-\Psi\Theta^{-1}\Phi}<\varepsilon,\quad \text{if } m\geq 2, \end{aligned} \tag{24}$$

i.e., the upper bound of the approximation error whose value is normalized to 1 at $k=0$.

(d) Construct the km-th order model $(\mathcal{A}_k,\mathcal{B}_k,\mathcal{C}_k)$.

B. Numerical Example

We consider a dynamical network over the square lattice composed of 25×40 nodes, as shown in Fig. 4. Suppose the dynamical network (A,B) with C is given as follows: $A\in\mathbb{R}^{1000\times 1000}$ is given by

$$\begin{aligned} &a_{i,j}=\left\{\begin{array}{l}1\quad(\text{if node } i \text{ and } j,\ i\neq j,\ \text{are connected})\\ 0\quad(\text{else, } i\neq j)\end{array}\right.\\ &r_1=1,\quad r_i=0,\quad i\neq 1 \end{aligned}$$

and $B\in\mathbb{R}^{1000\times 2}$ and $C\in\mathbb{R}^{1\times 1000}$ are given by

$$B=\left[\begin{array}{cccc}1&0&\cdots&0\\0&\cdots&0&-1\end{array}\right]^{\mathsf{T}},\ C=\left[\begin{array}{ccc}1&\cdots&1\end{array}\right],$$

i.e., the inputs are applied at the first and 1000-th nodes and the output is the sum of the all states.

The **RD**-realization $(\mathcal{A},\mathcal{B})$ with $\mathcal{C}$ of this system is given by Theorem 1. For the matrices $\Theta\in\mathbb{R}^{500\times 500}$ and $\Psi\in\mathbb{R}^{1\times 500}$ derived from (19) and (23), we denote the (i,j)-entry of Θ by $\Theta_{i,j}$ and the i-th entry of Ψ by Ψ_i. Then, in the upper figure of Fig. 5, for each k in the horizontal axis, the broken line shows the values of $\sum_{i=1}^{500}\Theta_{k,i}$ (i.e., reaction term), the line of $*$ shows the values of $\Theta_{k,k+1}$ ($=\phi_k$ in (19), i.e., diffusion term), and the solid line shows the values of Ψ_k. We can see from this figure that the entries of Ψ, corresponding the output matrix, have the value of almost

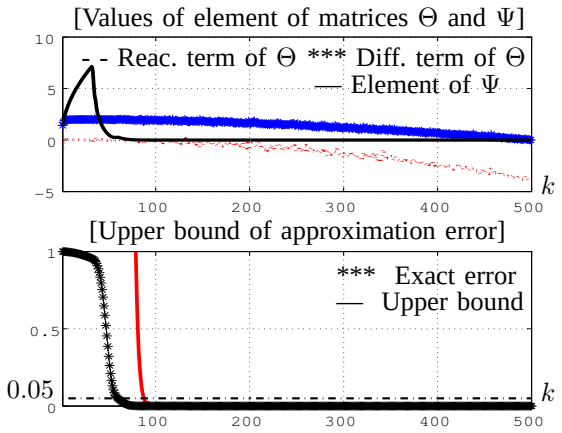

Fig. 5. Plots of matrices Θ and Ψ, and upper bound of the approximation error for dynamical network on two-dimensional lattice

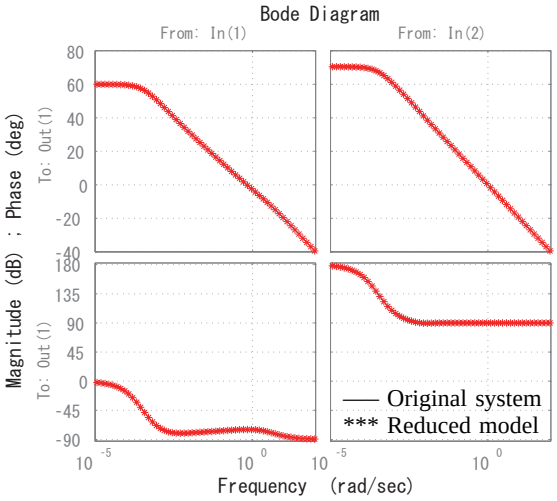

Fig. 6. Bode diagrams of dynamical network on two-dimensional lattice and reduced order model

zero for around $k \geq 60$, and each value of the reaction term and the diffusion term decreases as k increases.

The lower figure of Fig. 5 expresses the values of the left-hand side of (24) by the solid line, i.e., the upper bound of the exact error via the model order reduction, for each k. Furthermore, to validate the conservativeness of the error evaluation, the value of the *exact error* normalized to 1 at $k = 0$ is denoted by the line of $*$. This figure shows that (22) appropriately gives an upper bound of the approximation error. This figure also shows that the approximation error due to truncating $\mathcal{X}_{2\times 87}$ to $\mathcal{X}_{2\times 500}$ is small [4]. Then, the minimum of k satisfying (24) is given by $k = 87$ when $\varepsilon = 0.05$. From Fig. 6, which shows the Bode diagram of the original dynamical network (1000-th order; solid line) and that of the reduced order model (2×87-th order; line of $*$) at $k = 87$, we can see that both Bode diagrams are almost identical.

[4]The differential coefficient for the estimation is approximately calculated by using (21) at $\Delta\tau = 0.05$.

IV. Conclusion

In this paper, we proposed a model order reduction method for MIMO linear dynamical networks based on the Reaction-Diffusion transformation. The model reduction is performed by the direct truncation of the Reaction-Diffusion realization and preserves the stability of the system. The resultant approximation error is analyzed in terms of the H_∞-norm. Thanks to the numerical efficiency of the Reaction-Diffusion transformation, this method can be applied even to large-scale systems.

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