Network Clustering for SISO Linear Dynamical Networks via Reaction-Diffusion Transformation ⋆

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Abstract: In this paper, we study a network clustering problem for SISO linear dynamical networks. The proposed clustering method aggregates states which behave similarly for arbitrary input signals. We show that such states can efficiently be found via the *Reaction-Diffusion transformation*. The results give a model reduction procedure that preserves the stability and network structure.

Keywords: Network clustering, State aggregation, Large-scale systems, Model reduction, Network analysis

1. INTRODUCTION

Dynamical systems on large-scale complex networks, whose behaviors are determined by the interaction of a large number of subsystems, have been widely studied over the past decades. Examples of such dynamical networks include World-Wide-Web, gene regulatory networks, spread of infection; see [Boccalettia et al. (2006); Masuda and Konno (2010); Mesbahi and Egerstedt (2010)] for an overview. For such dynamical networks, it is often crucial to consider models whose state variables are *grouped*. In this paper, network clustering denotes the model reduction of such systems preserving the group structure; see Definition 3 for the precise definition. The goal of this paper is to propose a network clustering method, which is useful to capture coarse behavior of large-scale systems for both analysis and synthesis.

In existing literature, the state aggregation based on singular perturbation of dynamical networks have been intensively developed in [Aoki (1968); Chow and Kokotovic (1985); Biyik and Arcak (2006)]. However, this kind of approach cannot explicitly take account of the effect of the external input. There are also the other structure-preserving model reduction methods. The papers [S.Lall et al. (2003); Li and Bai (2006)] address the reduction preserving some underlying structure such as the Lagrangian structure and the second-order structure. However, these methods only deal with the preservation of certain block structure in the system matrices. In addition, although [Sandberg and Murray (2009)] discuss the reduction problem of each subsystem interconnected by a network, it requires a priori knowledge on clustering of the subsystems ([Yeung et al. (2009)] has somewhat relaxed the assumption). Egerestdt in [Egerstedt (2010)] has also solved a similar problem from the controllability and graph theory points of view for a limited class of linear dynamical networks.

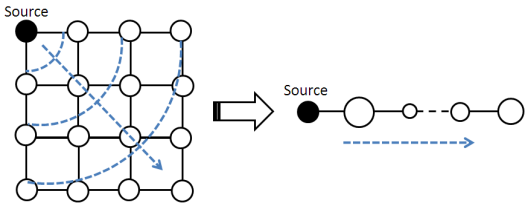

Fig. 1. Regarding diffusion on network as one-dimensional diffusion

On the other hand, this paper develops a clustering method by focusing on the input-to-state mapping of dynamical networks. This method does not need a priori knowledge of the partition of subsystems and approximates the input-to-state mapping of systems. In the method, we consider clustering the sets of nodes whose states have similar behavior for *arbitrary* input signals. Such nodes are efficiently found by using the network structure transformation called Reaction-Diffusion transformation, which has been proposed in [Ishizaki et al. (2010)] by the authors. The transformed system represents signal transmission of external input (diffusion source) over the network in the form of spatially *one-dimensional* reaction-diffusion, as shown in Fig. 1. Thanks to the numerical efficiency of the transformation, the network clustering method can be applied even for large-scale dynamical networks.

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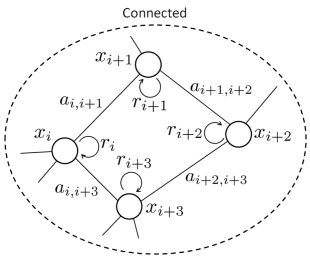

Fig. 2. Illustration of network structure

This paper is organized as follows: In Section 2, we describe a system to be studied and introduce some properties of the Reaction-Diffusion transformation. In Section 3, we solve a network clustering problem via the Reaction-Diffusion transformation. Section 4 shows the validity of the method by numerical examples. Section 5 concludes this paper.

NOTATION: For a vector v and a matrix $M = \{m_{ij}\}$, the following notation is used in this paper:

$\mathbb{R}$	the set of real numbers
I_n	the unit matrix of the size $n \times n$
e_k^n	the k-th column vector of I_n
$e_{k_1:k_2}^n$	the k_1-th to k_2-th columns of I_n
$\mathrm{diag}\,(M_1, \cdots, M_n)$	the block diagonal matrix composed of $M_1, \cdots, M_n$
$\mathrm{abs}\,(M)$	the matrix formed by $\{\lvert m_{ij}\rvert\}$

The H_∞-norm of a stable rational transfer matrix $G(s)$ is defined by

$$\|G(s)\|_\infty := \sup_{\omega \in \mathbb{R}} \sigma_{\max}\left(G(j\omega)\right).$$

Let $\mathcal{I}$ be the set of integers, for which $|\mathcal{I}|$ denotes the cardinality of $\mathcal{I}$ and $e_{\mathcal{I}}^n \in \mathbb{R}^{n \times |\mathcal{I}|}$ denotes the matrix whose column vectors are composed of e_k^n for $k \in \mathcal{I}$ (in some order of k), i.e., $e_{\mathcal{I}}^n = \left[e_{k_1}^n, \cdots, e_{k_m}^n\right] \in \mathbb{R}^{n \times m}$ for $\mathcal{I} = \{k_1, \cdots, k_m\}$.

2. REACTION-DIFFUSION TRANSFORMATION

In this paper, we deal with linear systems on large-scale complex networks whose general form is given as follows:

Definition 1. The linear system

$$\dot{x} = Ax + Bu, \quad x(0) = 0 \tag{1}$$

with $A = \{a_{i,j}\} \in \mathbb{R}^{n \times n}$ and $B = \{b_i\} \in \mathbb{R}^{n \times 1}$ is said to be a *dynamical network* (A, B) if A is stable and symmetric.

This is a generalization of *undirected reaction-diffusion* systems depicted in Fig. 2:

$$\dot{x}_i = -r_i x_i + \sum_{j=1, j \neq i}^{n} a_{i,j}\left(x_j - x_i\right) + b_i u \tag{2}$$

where $r_i (\geq 0)$ denotes the intensity of the reaction (chemical dissolution) of x_i, and $a_{i,j} (\geq 0)$ denotes the intensity of the diffusion between x_i and x_j. This coupled dynamics is stable if at least one r_i is strictly positive and the graph is connected. See, e.g., [Mesbahi and Egerstedt (2010)] for a survey on networked systems and multi-agent systems. This reaction-diffusion structure over the network can be represented in the following *spatially one-dimensional* manner:

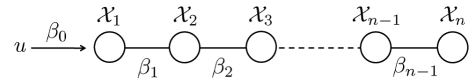

Fig. 3. Illustration of **RD**-realization

Definition 2. Let (A, B) a dynamical network in (1). Then, unitary H is said to be a *Reaction-Diffusion transformation matrix* if $\mathcal{A} := HAH^{\mathsf{T}}$ and $\mathcal{B} := HB$ are in the form of

$$\mathcal{A} = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{bmatrix} \in \mathbb{R}^{n \times n}, \ \mathcal{B} = \begin{bmatrix} \beta_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n \times 1} \tag{3}$$

with some negative constant $\alpha_i \in \mathbb{R}$ for $i \in \{1, \cdots, n\}$ and some non-negative constant $\beta_i \in \mathbb{R}$ for $i \in \{0, \cdots, n-1\}$. Moreover, the realization $(\mathcal{A}, \mathcal{B})$ is called a *Reaction-Diffusion realization* [1].

As shown in Theorem 1 in [Ishizaki et al. (2010)], we can effectively construct a **RD**-transformation matrix H. Moreover

$$\bar{i} := \begin{cases} \min\limits_i \{i : \beta_i = 0\} \text{ if } \prod_{i=1}^{n-1} \beta_i = 0, \\ n, \text{ otherwise} \end{cases} \tag{4}$$

does not depend on the choice of H. Actually, the column vectors of $H^{\mathsf{T}} e_{1:\bar{i}}^n$ span the controllable sub-space. That is, $\bar{i} = n$ if and only if (A, B) is controllable.

In what follows, we denote

$$g(s) := (sI_n - A)^{-1} B, \quad \begin{cases} \mathcal{G}(s) := (sI_n - \mathcal{A})^{-1} \mathcal{B} \\ \mathcal{G}_i(s) := (e_i^n)^{\mathsf{T}} \mathcal{G}(s). \end{cases} \tag{5}$$

We introduce a *low-pass* property equipped with the **RD**-realization.

Proposition 1. (Ishizaki et al. (2010)). Let $(\mathcal{A}, \mathcal{B})$ be the **RD**-realization in (3). Then, $\mathcal{G}_i$ in (5) satisfies

$$\|\mathcal{G}_i(s)\|_\infty = \mathcal{G}_i(0), \forall i \in \{1, \cdots, n\}. \tag{6}$$

Remark 1. The **RD**-transformation has a mathematically similar background of the Arnoldi algorithm in the Krylov projection [Antoulas (2005a,b); Gugercin (2002)]. However, the Krylov projection in general does not produce a tri-diagonal matrix with *non-negative off-diagonal entries*. On the other hand, we have proposed in [Ishizaki et al. (2010)] to make the off-diagonal entries non-negative by using the Householder transformation. Consequently, the non-negativity provides the uniqueness of the transformation and some preferable properties such as (6).

Example 1. Consider the simple dynamical network (A, B) given by

$$A = \begin{bmatrix} -7 & 1 & 2 & 2 & 1 \\ 1 & -1 & & & \\ 2 & & -2 & & \\ 2 & & & -2 & \\ 1 & & & & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

and the state vector $x := \left[x_1 \cdots x_5\right]^{\mathsf{T}}$. The network structure is depicted in Fig. 4.

Transforming (7) to the **RD**-realization, we have $\mathcal{A}, \mathcal{B}$ and $H \in \mathbb{R}^{5 \times 5}$ as

[1] The term "Reaction-Diffusion" is as necessary denoted as "**RD**-".

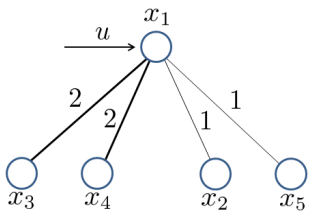

Fig. 4. Network structure of dynamical network

$$\mathcal{A} = \left[\begin{array}{ccc|cc} -7 & 3.16 & & & \\ 3.16 & -1.8 & 0.4 & & \\ & 0.4 & -1.2 & 0 & \\ \hline & & 0 & -1.2 & 0.4 \\ & & & 0.4 & -1.8 \end{array}\right], \quad \mathcal{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.316 & 0.633 & 0.633 & 0.316 \\ 0 & 0.633 & -0.316 & -0.316 & 0.633 \\ 0 & 0.633 & -0.316 & 0.316 & -0.633 \\ 0 & 0.316 & 0.633 & -0.633 & -0.316 \end{bmatrix}.$$

From the form of $\mathcal{A}$ and the definition of $\bar{i}$ in (4), we have $\bar{i} = 3$ and

$$H^{\mathsf{T}} e^n_{1:\bar{i}} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 0.316 & 0.633 \\ 0 & 0.633 & -0.316 \\ 0 & 0.633 & -0.316 \\ 0 & 0.316 & 0.633 \end{array}\right], \tag{8}$$

whose column vectors span the controllable sub-space of (A, B).

3. APPLICATION TO NETWORK CLUSTERING

3.1 Network Clustering based on State Aggregation

In this subsection, we outline a network clustering method based on state aggregation. First, we define the following notion of a network clustering:

Definition 3. Consider the dynamical network (A, B) in (1). A family of index sets $\{\mathcal{I}_{[l]}\}_{l\in\mathbb{L}}$ for $\mathbb{L} := \{1, \cdots, L\}$ is called a *cluster set* (its element is referred to as a cluster) if each element is a disjoint subset of $\{1, \cdots, n\}$ and $\bigcup_{l\in\mathbb{L}} \mathcal{I}_{[l]} = \{1, \cdots, n\}$. An *aggregation matrix* (*compatible with* $\{\mathcal{I}_{[l]}\}_{l\in\mathbb{L}}$) is defined by

$$\mathsf{P} := \operatorname{diag}\left(\mathsf{p}_{[1]}, \cdots, \mathsf{p}_{[L]}\right) Q \in \mathbb{R}^{L\times n} \tag{9}$$

with $\mathsf{p}_{[l]} \in \mathbb{R}^{1\times|\mathcal{I}_{[l]}|}$ such that $\mathsf{p}_{[l]}\mathsf{p}_{[l]}^{\mathsf{T}} = 1$, and the permutation matrix

$$Q = \left[\, e^n_{\mathcal{I}_{[1]}} \;\cdots\; e^n_{\mathcal{I}_{[L]}} \,\right]^{\mathsf{T}} \in \mathbb{R}^{n\times n}, e^n_{\mathcal{I}_{[l]}} \in \mathbb{R}^{n\times|\mathcal{I}_{[l]}|}. \tag{10}$$

Then, *the aggregated model* (*associated with* P) of the dynamical network (A, B) in (1) is given by

$$\left(\mathsf{P}A\mathsf{P}^{\mathsf{T}}, \mathsf{P}B\right). \tag{11}$$

In this definition, there are L clusters labeled by $l \in \mathbb{L}$. Then, each node belongs to exactly one of them, or equivalently, the behavior of l-th cluster is represented by $x_{[l]} := (e^n_{\mathcal{I}_{[l]}})^{\mathsf{T}} x \in \mathbb{R}^{|\mathcal{I}_{[l]}|}$. On the other hand, the aggregated model has the same number of clusters with the the scaler state variable $\hat{x}_{[l]} = \mathsf{p}_{[l]} x_{[l]} \in \mathbb{R}$. Note that $\mathsf{P}A\mathsf{P}^{\mathsf{T}}$ is symmetric and the aggregation matrix P clearly satisfies $\mathsf{P}\mathsf{P}^{\mathsf{T}} = I_L$.

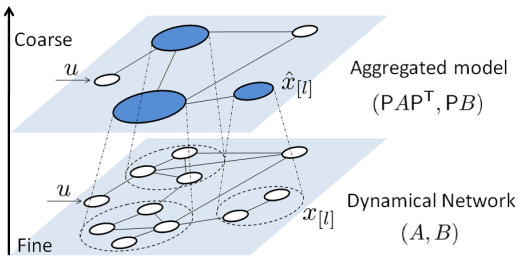

Fig. 5. Illustration of network clustering based on state aggregation

The aggregated model is constructed by applying the Petrov-Galerkin projection defined by $\Pi = \mathsf{P}^{\mathsf{T}}\mathsf{P}$ to the original dynamical network (A, B) (see Chapter 11 in [Antoulas (2005a)] for the detail.). In what follows, we derive a condition under which $x_{[l]} \in \mathbb{R}^{|\mathcal{I}_{[l]}|}$ can be recovered from $\hat{x}_{[l]} \in \mathbb{R}$ in a suitable sense.

Remark 2. Traditional model reduction methods such as the balanced truncation, the Krylov projection and the Hankel-norm approximation [Antoulas (2005a)] focus on approximating the input-to-output mapping of systems. As a result, the reduced model does not preserve the *spatial information* of the original systems. In other words, the network structure of the system is destroyed through the reduction. This drawback arises from the fact that every state in the reduced model in general contains information of all the states. On the contrary, we propose a model reduction based on the state aggregation of *local* states. As shown in Fig. 5, the network structure (spatial distribution) of internal states is retained through the reduction. We refer to such state aggregation of dynamical networks as *network clustering*.

Hereafter, we denote the transfer function of the aggregated model by

$$\mathsf{g}(s) := \mathsf{P}^{\mathsf{T}}\left(sI_L - \mathsf{P}A\mathsf{P}^{\mathsf{T}}\right)^{-1}\mathsf{P}B. \tag{12}$$

Then, let us begin with the simple situation where some of the original clusters have redundancy in the following sense:

Definition 4. Under Definition 3, if there exists a row-fullrank matrix $\mathsf{q}_{[l]} \in \mathbb{R}^{(|\mathcal{I}_{[l]}|-1)\times|\mathcal{I}_{[l]}|}$ such that

$$\mathsf{q}_{[l]}\left(e^n_{\mathcal{I}_{[l]}}\right)^{\mathsf{T}} g(s) = 0, \tag{13}$$

then the cluster $\mathcal{I}_{[l]}$ is said to be *reducible*.

The following theorem characterizes the reducibility of $\mathcal{I}_{[l]}$ via **RD**-transformation:

Theorem 1. Consider the **RD**-transformation of the dynamical network (1). Define

$$\mathsf{H}^{\bar{i}}_{[l]} := \left(e^n_{1:\bar{i}}\right)^{\mathsf{T}} H e^n_{\mathcal{I}_{[l]}} \in \mathbb{R}^{\bar{i}\times|\mathcal{I}_{[l]}|} \tag{14}$$

for $\bar{i}$ in (4). Then, (13) is equivalent to [2]

$$\mathsf{q}_{[l]}\left(\mathsf{H}^{\bar{i}}_{[l]}\right)^{\mathsf{T}} = 0. \tag{15}$$

Furthermore, for each $l \in \mathbb{L}$, take

$$\mathsf{p}_{[l]} := \frac{\hat{p}_{[l]}}{\left\|\hat{p}_{[l]}\right\|} \in \mathbb{R}^{1\times|\mathcal{I}_{[l]}|}, \quad \hat{p}_{[l]} := \left(\mathsf{H}^{\bar{i}}_{[l]} e^{\bar{i}}_1\right)^{\mathsf{T}} \mathsf{H}^{\bar{i}}_{[l]}. \tag{16}$$

[2] We allow empty $\mathsf{q}_{[l]}$.

Then, the aggregated model associated with P in (9) is stable and

$$g(s) = \mathsf{g}(s) \tag{17}$$

holds.

Proof. [Necessity of (15)] We have

$$\mathsf{q}_{[l]} \left(e^n_{\mathcal{I}_{[l]}}\right)^{\mathsf{T}} g(s) = \mathsf{q}_{[l]} \left(e^n_{\mathcal{I}_{[l]}}\right)^{\mathsf{T}} H^{\mathsf{T}} \mathcal{G}(s).$$

The desired result follows from the fact that $\mathcal{G}_i \equiv 0$ for $i > \bar{i}$, and $\{\mathcal{G}_i\}_{i=1}^{\bar{i}}$ are linearly independent.

[Sufficiency of (15)] Denoting

$$\left(e_i^{|\mathcal{I}_{[l]}|-1}\right)^{\mathsf{T}} \mathsf{q}_{[l]} = \left[\mathsf{q}^i_{1[l]}, \cdots, \mathsf{q}^i_{|\mathcal{I}_{[l]}|[l]}\right] \in \mathbb{R}^{1\times|\mathcal{I}_{[l]}|}$$

$$He^n_{\mathcal{I}_{[l]}} = \begin{bmatrix} \mathsf{h}^1_{1[l]} & \cdots & \mathsf{h}^1_{|\mathcal{I}_{[l]}|[l]} \\ \vdots & \cdots & \vdots \\ \mathsf{h}^n_{1[l]} & \cdots & \mathsf{h}^n_{|\mathcal{I}_{[l]}|[l]} \end{bmatrix} \in \mathbb{R}^{n\times|\mathcal{I}_{[l]}|},$$

the condition (15) implies

$$\sum_{j=1}^{|\mathcal{I}_{[l]}|} \mathsf{q}^i_{j[l]} \mathsf{h}^k_{j[l]} = 0, \quad \forall \begin{cases} i \in \{1, \cdots, |\mathcal{I}_{[l]}| - 1\} \\ k \in \{1, \cdots, \bar{i}\}. \end{cases}$$

By Proposition 1, we have

$$\begin{aligned} &\left\| \left(e_i^{|\mathcal{I}_{[l]}|-1}\right)^{\mathsf{T}} \mathsf{q}_{[l]} \left(e^n_{\mathcal{I}_{[l]}}\right)^{\mathsf{T}} g(s) \right\|_\infty \qquad (18) \\ &= \left\| \sum_{k=1}^{n} \sum_{j=1}^{|\mathcal{I}_{[l]}|} \mathsf{q}^i_{j[l]} \mathsf{h}^k_{j[l]} \mathcal{G}_k(s) \right\|_\infty \le \sum_{k=1}^{n} \left| \sum_{j=1}^{|\mathcal{I}_{[l]}|} \mathsf{q}^i_{j[l]} \mathsf{h}^k_{j[l]} \right| \mathcal{G}_k(0) \end{aligned}$$

where the last term is 0 from the fact that $\mathcal{G}_i \equiv 0$ for $i > \bar{i}$.

[Proof of (17)] The stability of $\mathsf{g}(s)$ is trivial from the negative definiteness of A. Note that $\mathcal{I}_{[l]}$ is reducible if and only if the (column) rank of $\mathsf{H}^{\bar{i}}_{[l]}$ is 1. This is equivalent to that there exist $c_j \in \mathbb{R}$ such that $\mathsf{H}^{\bar{i}}_{[l]} e^{\bar{i}}_j = c_j \mathsf{H}^{\bar{i}}_{[l]} e^{\bar{i}}_1$ for all j. Here, $\hat{p}_{[l]} \neq 0$ holds from $\mathsf{H}^{\bar{i}}_{[l]} e^{\bar{i}}_1 \neq 0$. Then, we have

$$\mathsf{H}^{\bar{i}}_{[l]} = \frac{\mathsf{H}^{\bar{i}}_{[l]} e^{\bar{i}}_1}{\left(\mathsf{H}^{\bar{i}}_{[l]} e^{\bar{i}}_1\right)^{\mathsf{T}} \mathsf{H}^{\bar{i}}_{[l]} e^{\bar{i}}_1} \hat{p}_{[l]}.$$

This implies that $(\mathsf{p}_{[l]})^{\mathsf{T}} \mathsf{q}_{[l]} = 0$ holds for $\mathsf{q}_{[l]}$ satisfying (15). Here, considering the coordinate transformation by some unitary matrix $[\mathsf{P}^{\mathsf{T}}, \overline{\mathsf{P}}^{\mathsf{T}}]^{\mathsf{T}}$, we have

$$\begin{aligned} g(s) &= \mathsf{g}(s) + \Xi(s)\, \overline{\mathsf{P}} (sI_n - A)^{-1} B \qquad (19) \\ \Xi(s) &= \mathsf{P}^{\mathsf{T}} \left(sI_L - \mathsf{P}A\mathsf{P}^{\mathsf{T}}\right)^{-1} \mathsf{P}A\overline{\mathsf{P}}^{\mathsf{T}} + \overline{\mathsf{P}}^{\mathsf{T}}. \end{aligned}$$

Note that $\Xi(s)$ is stable. Define $\overline{\mathsf{P}}$ by replacing $\mathsf{p}_{[l]}$ with $\mathsf{q}_{[l]}$ for each $l \in \mathbb{L}$ in (9). Then, $[\mathsf{P}^{\mathsf{T}}, \overline{\mathsf{P}}^{\mathsf{T}}]^{\mathsf{T}}$ is unitary, and $\overline{\mathsf{P}} (sI_n - A)^{-1} B = 0$ by the reducibility of (13). Hence, the result follows.

This theorem characterizes the reducibility of $\mathcal{I}_{[l]}$ in (13) via the (column) rank deficiency of $\mathsf{H}^{\bar{i}}_{[l]}$ in (14), which is obtained through the **RD**-transformation. Note that the rank of $\mathsf{H}^{\bar{i}}_{[l]}$ satisfying (15) is 1. This means that $\mathsf{H}^{\bar{i}}_{[l]}$ is composed of column vectors having an *identical* direction. Moreover, for a dynamical network with the output mapping of C, we obtain that of the aggregated model by $C\mathsf{P}^{\mathsf{T}}$.

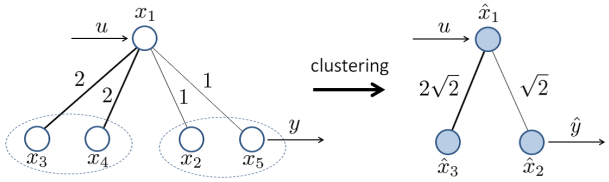

Fig. 6. Illustration of network clustering

Example 2. (continuation of Example 1)

Consider the dynamical network (A, B) dealt with in Example 1. Furthermore, consider the output mapping of

$$C = [0, 0, 0, 0, 1].$$

By (8), we have

$$\left(e^n_{1:\bar{i}}\right)^{\mathsf{T}} H = \left[\begin{array}{c|cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.316 & 0.633 & 0.633 & 0.316 \\ 0 & 0.633 & -0.316 & -0.316 & 0.633 \end{array}\right].$$

From this matrix, we can see that the second and fifth column vectors and the third and fourth column vectors are respectively identical. This implies that $\mathsf{H}^{\bar{i}}_{[l]}$ in (14) has column rank 1 with the sets

$$\mathcal{I}_{[1]} = \{1\}, \quad \mathcal{I}_{[2]} = \{2, 5\}, \quad \mathcal{I}_{[3]} = \{3, 4\}.$$

Therefore, $\{\mathcal{I}_{[l]}\}_{l\in\{1,2,3\}}$ is a reducible cluster set. Then, by Theorem 1, we have

$$\mathsf{p}_{[1]} = 1, \ \mathsf{p}_{[2]} = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], \ \mathsf{p}_{[3]} = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right].$$

By using these vectors for (9), we obtain the aggregated model satisfying (17) as

$$\mathsf{P}A\mathsf{P}^{\mathsf{T}} = \begin{bmatrix} -7 & \sqrt{2} & 2\sqrt{2} \\ \sqrt{2} & -1 & \\ 2\sqrt{2} & & -2 \end{bmatrix}, \ \mathsf{P}B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ C\mathsf{P}^{\mathsf{T}} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}^{\mathsf{T}}.$$

This network structure transformation is depicted in Fig. 6.

Remark 3. The network clustering method proposed in this subsection is based on eliminating the uncontrollable sub-space of dynamical networks (A, B) via the aggregation matrix in (9). Since the aggregation matrix has the block-diagonal structure, this method is different from the *simple* elimination of the uncontrollable sub-space.

3.2 Approximation of Controllable Sub-Space

In this subsection, aiming at more significant order reduction, we relax the reducibility of $\mathcal{I}_{[l]}$ via its equivalent characterization of (15). Here, let us consider replacing $\bar{i}$ in (15) and (16) with some $k(\le \bar{i})$ for relaxing the reducibility condition. Recall that Theorem 1 provides the *exact* reduction as in (17) thanks to the property that $\mathcal{G}_i \equiv 0$ for $i > \bar{i}$. This property stems from the fact that the $\bar{i}$ column vectors of $H^{\mathsf{T}} e^n_{1:\bar{i}}$ span the controllable sub-space of (A, B). From the perspective of relaxation, we could expect small error $\|g(s) - \mathsf{g}(s)\|$ by replacing $\bar{i}$ with

$k(\leq \bar{i})$ as long as the column vectors of $H^{\mathsf{T}} e^n_{1:k}$ span a space which appropriately approximates the controllable sub-space of (A, B).

Let $\mathsf{h}^k_{j[l]}$ for $j \in \{1, \cdots, |\mathcal{I}_{[l]}|\}$ denote a column vector of $\mathsf{H}^k_{[l]}$, namely

$$\mathsf{H}^k_{[l]} = \left[\mathsf{h}^k_{1[l]}, \cdots, \mathsf{h}^k_{|\mathcal{I}_{[l]}|[l]}\right], \quad \mathsf{h}^k_{j[l]} \in \mathbb{R}^{k\times 1}. \tag{20}$$

Then, the condition $\mathrm{rank}(\mathsf{H}^{\bar{i}}_{[l]}) = 1$ can be rewritten by

$$\left|\left(\mathsf{h}^k_{1[l]}\right)^{\mathsf{T}} \mathsf{h}^k_{j[l]}\right| \geq (1-\varepsilon)\left\|\mathsf{h}^k_{1[l]}\right\|\left\|\mathsf{h}^k_{j[l]}\right\|, \quad j \in \{1, \cdots, |\mathcal{I}_{[l]}|\} \tag{21}$$

with $k = \bar{i}$ and $\varepsilon = 0$. Therefore, we consider determining a cluster $\mathcal{I}_{[l]}$ satisfying (21) for some $k(\leq \bar{i})$ under a preassigned constant $\varepsilon \ll 1$. The following proposition provides a valid measure to determine k, which can be regarded as the dimension of approximated controllable sub-space:

Proposition 2. (Ishizaki et al. (2010)). Consider the **RD**-realization $(\mathcal{A}, \mathcal{B})$ with the output matrix $\mathcal{C} \in \mathbb{R}^{1\times n}$. Denote

$$G(s) = \mathcal{C}(sI_n - \mathcal{A})^{-1}\mathcal{B}, \quad \hat{G}_k(s) = \mathcal{C}_k(sI_n - \mathcal{A}_k)^{-1}\mathcal{B}_k$$

where

$$\mathcal{A}_k := (e^n_{1:k})^{\mathsf{T}} \mathcal{A} e^n_{1:k}, \quad \mathcal{B}_k := (e^n_{1:k})^{\mathsf{T}}\mathcal{B}, \quad \mathcal{C}_k := \mathcal{C} e^n_{1:k}.$$

If $\mathcal{A}$ is stable, then $\mathcal{A}_k$ is stable and

$$\left\|G(s) - \hat{G}_k(s)\right\|_\infty \leq \left|\mathrm{abs}(\mathcal{C})\mathcal{A}^{-1}\mathcal{B} - \mathrm{abs}(\mathcal{C}_k)\mathcal{A}_k^{-1}\mathcal{B}_k\right| \tag{22}$$

holds. Moreover, if the entries of $\mathcal{C}$ have the same sign, (22) holds with equality. ■

This proposition shows that the resultant error due to the *direct truncation* of the **RD**-realization with k-th order is bounded by the right-hand side of (22). This implies that a k-dimensional sub-space spanned by the column vectors $H^{\mathsf{T}} e^n_{1:k}$ approximates the controllable sub-space of (A, B) as long as the right-hand side of (22) is small enough. Therefore, we use such k to replace $\bar{i}$ in (16). The algorithm of this technique is as follows:

(a) Calculate the **RD**-transformation of the dynamical network (A, B) with C.
(b) By Proposition 2 with $\mathcal{C} := CH^{\mathsf{T}}$, find the minimum k such that

$$\frac{\mathrm{abs}(\mathcal{C}_k)\mathcal{A}_k^{-1}\mathcal{B}_k - \mathrm{abs}(\mathcal{C})\mathcal{A}^{-1}\mathcal{B}}{-\mathrm{abs}(\mathcal{C})\mathcal{A}^{-1}\mathcal{B}} \leq \gamma, \tag{23}$$

i.e., the upper bound of the normalized approximation error, for some preassigned $\gamma \ll 1$.
(c) Find a cluster set $\{\mathcal{I}_{[l]}\}_{l\in\mathbb{L}}$ satisfying (21) for some preassigned $\varepsilon \ll 1$.
(d) Derive the aggregation matrix P in (9) with (16) for which $\bar{i}$ is replaced with k.
(e) Construct the aggregated model $(\mathsf{P}A\mathsf{P}^{\mathsf{T}}, \mathsf{P}B)$ with $C\mathsf{P}^{\mathsf{T}}$.

Remark 4. The algorithm proposed here can well work; see the numerical example in Section 4. The analysis of the resulting error $\|g(s) - \mathsf{g}(s)\|$ will be reported in later publication.

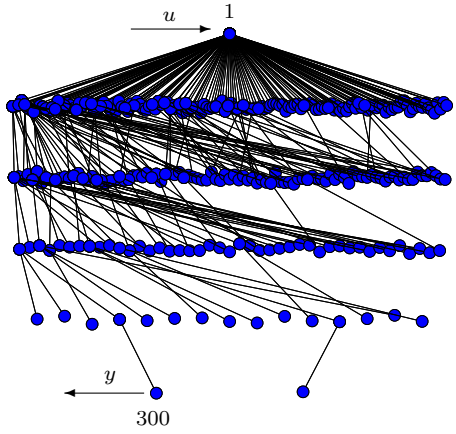

Fig. 7. Complex network of Barabasi-Albert model

4. NUMERICAL EXAMPLE OF A COMPLEX NETWORK REDUCTION

We apply the proposed algorithm in Section 3.2. Here, we consider a dynamical network on a Barabasi-Albert model (power of the preferential attachment: 1.2) shown in Fig. 7, which is well-known as a complex network model satisfying the scale-free and small-world property [Boccalettia et al. (2006)]. This network has 300 nodes and 300 edges in which some hubs are included and the first node is connected to every node within 5 edges. In this figure, each node is ordered accordingly to the distance (number of edges) from the first node.

The dynamical network (A, B) with C is given as follows: For (2), $A \in \mathbb{R}^{300\times 300}$ is given by

$$a_{i,j} = \begin{cases} 1 & (\text{if node } i \text{ and } j,\ i \neq j, \text{ are connected}) \\ 0 & (\text{else, } i \neq j) \end{cases}$$

$$r_1 = 1, \quad r_i = 0 \quad \forall i,\ i \neq 1,$$

and $B \in \mathbb{R}^{300\times 1}$ and $C \in \mathbb{R}^{1\times 300}$ are given by

$$B = [1, 0, \cdots, 0]^{\mathsf{T}}, \quad C = [0, \cdots 0, 1]$$

i.e., the input is applied at the first node and the output is the state of the 300-th node.

Computing the **RD**-transformation and finding minimum k satisfying (23) for $\gamma = 0.05$, we have $k = 26$. Then, finding a cluster set $\{\mathcal{I}_{[l]}\}_{l\in\mathbb{L}}$ satisfying (21) for $\varepsilon = 0.01$ and constructing the aggregation matrix in (9), we obtain an aggregated model with the 71-th order. Here, Fig. 8 shows the network structure of the aggregated model, and we can see from the figure that some nodes are aggregated into clusters.

Furthermore, Fig. 9 shows the Bode diagram of the original dynamical network (the solid line; 300-th order) and that of the aggregated model (the line of $*$; 71-th order). From this figure, we can see that both Bode diagrams are almost identical.

5. CONCLUSION

In this paper, a network clustering method for SISO linear dynamical networks has been proposed by using Reaction-Diffusion transformation. From the point of view of the control theory, the sets of states which behaves similarly

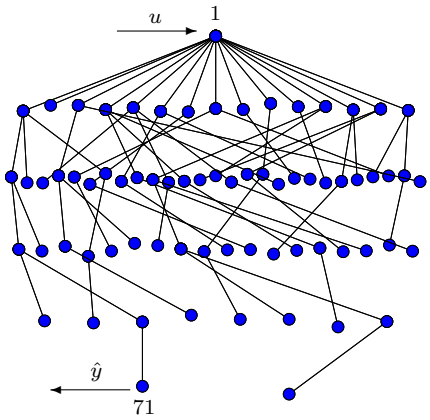

Fig. 8. Complex network of Barabasi-Albert model

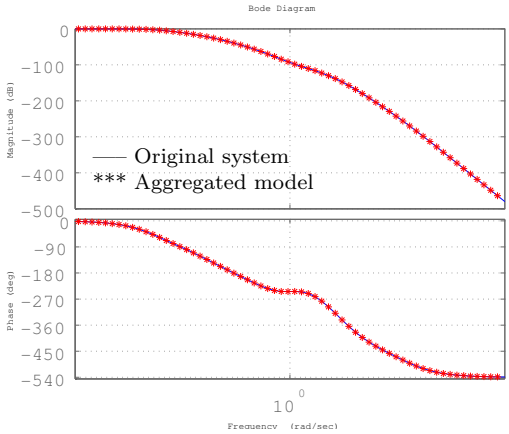

Fig. 9. Bode diagrams of original system and aggregated model

for arbitrary input signals are interpreted as sets of uncontrollable states. In addition, it has been shown that such states are efficiently found via Reaction-Diffusion transformation. The method provides the aggregated model preserving the stability and cluster structure of the original system. Consequently, the network structure is retained through the aggregation.

The error evaluation of aggregated models constructed by the technique in Section 3.2 is one of future work.

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