Model Reduction of Multi-Input Dynamical Networks based on Clusterwise Controllability

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***Abstract*— This paper proposes a model reduction method for a multi-input linear system evolving on large-scale complex networks, called dynamical networks. In this method, we construct a set of clusters (i.e., disjoint subsets of state variables) based on a notion of clusterwise controllability that characterizes a kind of local controllability of the state-space. The clusterwise controllability is determined through a basis transformation with respect to each input. Aggregating the constructed clusters, we obtain a reduced model that preserves interconnection topology of the clusters as well as some particular properties, such as stability, steady-state characteristic and system positivity. In addition, we derive an H_∞-error bound of the state discrepancy caused by the aggregation. The efficiency of the proposed method is shown by a numerical example including a large-scale complex network.**

I. Introduction

Dynamical systems on large-scale complex networks (large-scale dynamical networks), whose behavior is described by an interaction of a large number of interconnected subsystems, have been widely studied over the past decades. Examples of such dynamical networks include World-Wide-Web, gene regulatory networks, spread of infection; see [1], [2], [3], [4] for an overview. In general, due to their large-scale complex network topology, the straightforward application of traditional analysis and design methods is often unrealistic. Therefore, model reduction is indispensable for overcoming such a difficulty; see [5], [6], [7] for survey articles.

The balanced truncation, the Hankel-norm approximation and the Krylov projection, which are well-known as traditional model reduction methods, provide a reduced model suitably approximating the input-to-output mapping of a given system [5], [6]. However, these traditional reduction methods have a common drawback in applying to dynamical networks; The interconnection structure among the original states is lost through the reduction. More specifically, each state of the reduced model is usually obtained by the linear combination of all the states of the original system, namely the transformation matrix for reduction is *dense*. For dynamical networks, it is more important to address a network structure preserving model reduction problem. We attempt to address this problem by means of an input-to-*state* mapping approximation using transformation matrices on which a suitable *sparsity* is imposed.

As related studies, structure preserving mode reduction problems have been discussed in various literature [8], [9], [10]. However, these problems are not formulated based on the premise of the network structure preservation even though they consider the preservation of some underlying structure of systems, such as the Lagrangian structure and the second-order structure. Furthermore, the paper [8], where a network structure preserving model reduction problem is considered, does not discuss the relation between partition of subsystems and the resultant approximation error.

Against such a background, we have proposed in [11] a network structure preserving model reduction method (network clustering method) for single-input dynamical networks. In this method, we construct a set of clusters (i.e., disjoint subsets of state variables) based on a notion of cluster reducibility that coincides with a kind of local uncontrollability of the state-space of the dynamical networks. Aggregating the cluster set under suitable weighting, we obtain a reduced model that preserves interconnection topology of the original system as well as some specific properties, such as stability, steady-state characteristic and system positivity. In addition, we have derived an H_∞-error bound of the state discrepancy caused by the aggregation.

This paper extends the network clustering scheme for single-input systems [11] to that for multi-input systems. To this end, transforming the basis of its state-space with respect to each input, we derive a necessary and sufficient condition of the cluster reducibility, which is the converse concept of clusterwise controllability. This basis transformation is based on tri-diagonalization of the system matrices, whose fundamental properties are useful for constructing reducible clusters. Furthermore, we introduce more general formulation of weak cluster reducibility suitable for establishing a clustering scheme for the multi-input systems.

This paper is organized as follows: In Section II, we describe a linear system evolving on complex networks and formulate a network clustering problem for this class of systems. Furthermore, we define a basis transformation implemented as matrix tri-diagonalization that is important to solve the network clustering problem. In Section III, defining a notion of clusterwise controllability, we give a solution by fully exploiting the fundamental properties of the tri-diagonalization. Furthermore, we show the efficiency of the proposed method through a numerical example of a dynamical network with 1000 nodes and 2000 edges. Finally,

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Section IV concludes this paper.

NOTATION The following notation is to be used:

$\mathbb{R}$ ($\mathbb{R}_+$)	the set of (positive) real numbers
I_n	the unit matrix of the size $n \times n$
e_k^n	the kth column vector of I_n
$e_{k_1:k_2}^n$	the k_1th to k_2th columns of I_n
$\mathrm{span}\,(M)$	the space spanned by the column vectors of a matrix M
$\|M\| = \sigma_{\max}(M)$	the maximum singular value of a matrix M
$\mathrm{diag}\,(v)$	the diagonal matrix whose diagonal entries are the entries of a vector v
$\mathrm{Diag}\,(M_1, \ldots, M_n)$	the block-diagonal matrix composed of matrices $M_1, \ldots, M_n$

The H_∞-norm of a stable proper transfer matrix $G(s)$ is defined by $\|G(s)\|_\infty := \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(j\omega))$. For a set $\mathcal{I}$ of the natural numbers, $|\mathcal{I}|$ denotes the cardinality of $\mathcal{I}$, and $e_{\mathcal{I}}^n \in \mathbb{R}^{n \times |\mathcal{I}|}$ denotes the matrix whose column vectors are composed of e_k^n for $k \in \mathcal{I}$ (in some order of k), i.e., $e_{\mathcal{I}}^n = [e_{k_1}^n, \ldots, e_{k_m}^n] \in \mathbb{R}^{n \times m}$ for $\mathcal{I} = \{k_1, \ldots, k_m\}$.

II. Preliminary

A. Problem Formulation

This paper deals with a linear system evolving on large-scale complex networks whose general form is defined as follows:

Definition 1: A linear system

$$\dot{x} = Ax + Bu \tag{1}$$

with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ is said to be a *dynamical network* (A, B) if A is stable and symmetric. Moreover, if the off-diagonal entries of A and the entries of B are all non-negative, it is said to be a *positive* dynamical network.

Let us see the following (spatially-discrete) reaction-diffusion system evolving on a complex network, of particular interest in this paper:

$$\dot{x}_i = -r_i x_i + \sum_{j=1, j\neq i}^{n} a_{i,j}(x_j - x_i) + \sum_{k=1}^{m} b_{i,k} u_k \tag{2}$$

for $i \in \{1, \ldots, n\}$ where $r_i \geq 0$ denotes the intensity of the reaction (chemical dissolution) of x_i, and $a_{i,j} = a_{j,i}$ for $i \neq j$ denotes the intensity of the diffusion between x_i and x_j; see Fig. 1. This system is known as a basic model that represents a diffusion process evolving over complex networks (see [12]). The interaction topology of (2) can be represented in a graph theoretic fashion: By defining the reaction matrix $R := \mathrm{diag}([r_1, \cdots, r_n])$ and the symmetric weighted graph Laplacian $\mathcal{L} := \{l_{i,j}\}$ for

$$l_{i,j} = \begin{cases} -a_{i,j}, & i \neq j \\ \sum_{j=1, j\neq i}^{n} a_{i,j}, & i = j, \end{cases}$$

the system (2) is represented as a dynamical network with the state vector $x := [x_1, \ldots, x_n]^{\mathsf{T}} \in \mathbb{R}^n$ and

$$A = -(R + \mathcal{L}), \; B = [b_1, \ldots, b_m] \tag{3}$$

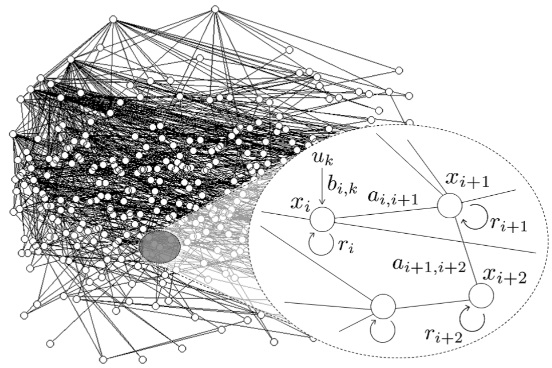

Fig. 1. Depiction of Dynamical Systems Evolving on Complex Network.

where $b_k := [b_{1,k}, \ldots, b_{n,k}]^{\mathsf{T}} \in \mathbb{R}^n$. This class of systems is stable if $r_i > 0$ holds for at least one i, $a_{i,j} \geq 0$ holds for all $i \neq j$, and $\mathcal{L}$ is irreducible.

In what follows, we formulate a network structure preserving model reduction problem (network clustering problem). First, the following notion of network clustering is defined:

Definition 2: Let (A, B) be a dynamical network. The family of an index set $\{\mathcal{I}_{[l]}\}_{l \in \mathbb{L}}$ for $\mathbb{L} := \{1, \ldots, L\}$ is called a *cluster set* (its element is referred to as a cluster) if each element $\mathcal{I}_{[l]}$ is a disjoint subset of $\{1, \ldots, n\}$ and it satisfies $\bigcup_{l \in \mathbb{L}} \mathcal{I}_{[l]} = \{1, \ldots, n\}$. Furthermore, an *aggregation matrix* compatible with $\{\mathcal{I}_{[l]}\}_{l \in \mathbb{L}}$ is defined by

$$\mathsf{P} := \mathrm{Diag}\left(\mathsf{p}_{[1]}, \ldots, \mathsf{p}_{[L]}\right) \Pi \in \mathbb{R}^{\Delta \times n}, \; \Delta := \sum_{l=1}^{L} \delta_l \tag{4}$$

with $\mathsf{p}_{[l]} \in \mathbb{R}^{\delta_l \times |\mathcal{I}_{[l]}|}$ such that $\delta_l \leq |\mathcal{I}_{[l]}|$ and $\mathsf{p}_{[l]} \mathsf{p}_{[l]}^{\mathsf{T}} = I_{\delta_l}$, and the permutation matrix

$$\Pi := [e_{\mathcal{I}_{[1]}}^n, \ldots, e_{\mathcal{I}_{[L]}}^n]^{\mathsf{T}} \in \mathbb{R}^{n \times n}, \; e_{\mathcal{I}_{[l]}}^n \in \mathbb{R}^{n \times |\mathcal{I}_{[l]}|}. \tag{5}$$

Then, the *aggregated model* of (A, B) associated with P is given by

$$\left(\mathsf{P} A \mathsf{P}^{\mathsf{T}}, \mathsf{P} B\right). \tag{6}$$

This aggregated model is constructed by projecting the controllable subspace of (A, B) onto $\mathrm{span}(\mathsf{P}^{\mathsf{T}})$. Note that $\mathsf{P}\mathsf{P}^{\mathsf{T}} = I_\Delta$ holds and $\mathsf{P} A \mathsf{P}^{\mathsf{T}}$ is stable and symmetric. For this network clustering, we give the following intuitive explanation: There are L clusters labeled by $l \in \mathbb{L}$, and each node of the original network belongs to exactly one of the clusters. On the other hand, the number of clusters in the aggregated model is the same as that in the original system. Let $x_{[l]} := (e_{\mathcal{I}_{[l]}}^n)^{\mathsf{T}} x \in \mathbb{R}^{|\mathcal{I}_{[l]}|}$ denote the state of the lth cluster. The linear transformation performed by the aggregation matrix (4) implies $\mathsf{x}_{[l]} = \mathsf{p}_{[l]} x_{[l]}$ for $l \in \mathbb{L}$ where $\mathsf{x}_{[l]} \in \mathbb{R}^{\delta_l}$ is an aggregated state of the cluster. We can see that this transformation represents the aggregation of the original state $x_{[l]} \in \mathbb{R}^{|\mathcal{I}_{[l]}|}$ into the aggregated one $\mathsf{x}_{[l]} \in \mathbb{R}^{\delta_l}$ under the weighting of $\mathsf{p}_{[l]} \in \mathbb{R}^{\delta_l \times |\mathcal{I}_{[l]}|}$. Consequently, the interconnection topology (spatial distribution) of the states is preserved through the reduction; see Fig. 2.

In what follows, we derive a condition that $\mathsf{x}_{[l]}$ behaves similarly to $x_{[l]}$ in a suitable sense. From a model reduc-

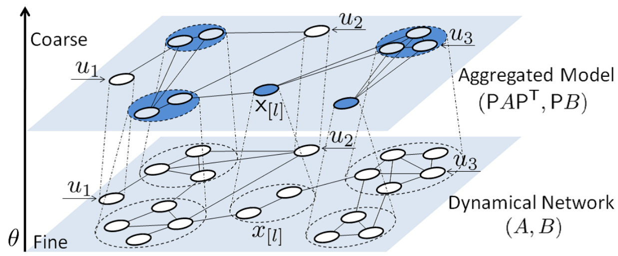

Fig. 2. Depiction of Network Clustering.

tion point of view, a smaller δ_l is desirable to reduce the dimension of dynamical networks.

Hereafter, the transfer function from the input to the all state variables in the dynamical network (1) and that of the aggregated model (6) are denoted by

$$\begin{aligned} g(s) &:= (sI_n - A)^{-1} B, \\ \hat{\mathsf{g}}(s) &:= \mathsf{P}^\mathsf{T} \left(sI_\Delta - \mathsf{P}A\mathsf{P}^\mathsf{T} \right)^{-1} \mathsf{P}B, \end{aligned} \tag{7}$$

respectively. Then, the network clustering problem to be addressed is formulated as follows:

Problem 1: Let (A, B) be a dynamical network. Given a constant $\epsilon \in \mathbb{R}_+$, find an aggregation matrix P in (4) compatible with $\{\mathcal{I}_{[l]}\}_{l \in \mathbb{L}}$ such that

$$\|g(s) - \hat{\mathsf{g}}(s)\|_\infty \le \epsilon.$$

B. Positive Tri-Diagonalization

In this paper, we make use of the following matrix tri-diagonalization to investigate the behavior of clusters:

Definition 3: Let $A \in \mathbb{R}^{n \times n}$ be a matrix and $b \in \mathbb{R}^n$ a vector. A unitary transformation by $H \in \mathbb{R}^{n \times n}$ is said to be *positive tri-diagonalization* of the pair $\langle A, b \rangle$ if $\mathfrak{A} := H^\mathsf{T} A H \in \mathbb{R}^{n \times n}$ and $\mathfrak{b} := H^\mathsf{T} b \in \mathbb{R}^n$ are in the form of

$$\mathfrak{A} = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{bmatrix}, \quad \mathfrak{b} = \begin{bmatrix} \beta_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{8}$$

with $\beta_i \ge 0$ for all $i \in \{1, \ldots, n-1\}$. Moreover, the pair $\langle \mathfrak{A}, \mathfrak{b} \rangle$ is referred to as a *positive tri-diagonal pair* of $\langle A, b \rangle$.

Note that this transformation is defined for the pair of a matrix A and a vector b. In terms of the dynamical network (A, B) having m inputs, the positive tri-diagonalization of $\langle A, b_k \rangle$, where $b_k \in \mathbb{R}^n$ denotes the kth column of $B \in \mathbb{R}^{n \times m}$, is exploited to investigate the behavior of the systems with respect to u_k. The following propositions show valuable properties of the positive tri-diagonalization (see, e.g., [5], [13] for the proofs):

Proposition 1: Let $A \in \mathbb{R}^{n \times n}$ be a matrix and $b \in \mathbb{R}^n$ a vector. For the pair $\langle A, b \rangle$, there exists a positive tri-diagonalizing matrix $H \in \mathbb{R}^{n \times n}$ that satisfies

$$\begin{aligned} &\mathrm{span}\left(H e^n_{1:\bar{i}} \right) = \mathrm{span}\left(\left[b, Ab, \ldots, A^{n-1} b \right] \right), \\ &\bar{i} := \begin{cases} \min\limits_i \{ i : \beta_i = 0 \}, & \text{if } \prod_{i=1}^{n-1} \beta_i = 0 \\ n, & \text{otherwise.} \end{cases} \end{aligned} \tag{9}$$

This proposition shows the existence condition of the positive tri-diagonalization and relates the range space of H to the $\bar{i}$th dimensional controllable sub-space of $\langle A, b \rangle$. Note that there exists a positive tri-diagonalizing matrix H_k for each $\langle A, b_k \rangle$ thanks to the supposition of A in Definition 1.

Proposition 2: Given a positive tri-diagonal pair $\langle \mathfrak{A}, \mathfrak{b} \rangle$ with a stable $\mathfrak{A}$, define $\mathfrak{G}^i(s) := (e^n_i)^\mathsf{T} (sI_n - \mathfrak{A})^{-1} \mathfrak{b}$. Then, the relative degree of $\mathfrak{G}^i$ is i, and

$$\left\| \mathfrak{G}^i(s) \right\|_\infty = \mathfrak{G}^i(0) \tag{10}$$

holds for all $i \in \{1, \ldots, n\}$.

This proposition shows that $\mathfrak{G}^i$ (i.e., the transfer function from the input to the ith state in the linear system $(\mathfrak{A}, \mathfrak{b})$) has the low-pass property represented by (10). This property is useful for deriving an H_∞-error bound in Section III.

Remark 1: The tri-diagonalization procedures of large matrices have been widely investigated in, e.g., numerical algebra community toward various applications, such as eigenvalue computations [5]. In particular for symmetric matrices, unitary transformations are often used to preserve the symmetry. It should be emphasized that the tri-diagonalization procedures do not require computationally expensive operations as discussed in various literature [5], [7], [14]. Especially for sparse matrices, such as the graph Laplacian of complex networks, the numerical efficiency stands out. In this sense, the positive tri-diagonalization can be implemented even for large-scale dynamical networks.

III. Network Clustering

A. Exactly Reducible Case

In order to address the network clustering problem, we define the following notion of local controllability:

Definition 4: Let be given a dynamical network (A, B) with the initial state $x(0) = 0$. Under Definition 2, the state of a cluster $\mathcal{I}_{[l]}$ is said to be *clusterwise controllable* if there exists an input $u_{[l]}(t) \in \mathbb{R}^m$ for $t \in [0, \bar{t}]$ such that $(e^n_{\mathcal{I}_{[l]}})^\mathsf{T} x(\bar{t}) = \bar{x}_{[l]}$ for some $\bar{t} > 0$ and any $\bar{x}_{[l]} \in \mathbb{R}^{|\mathcal{I}_{[l]}|}$.

The clusterwise controllability characterizes whether the state $x_{[l]}$ of a cluster $\mathcal{I}_{[l]}$ can be steered toward an arbitrary state $\bar{x}_{[l]}$ within a finite time interval. This clusterwise controllability coincides with a class of output controllability, where the output matrix is taken as $(e^n_{\mathcal{I}_{[l]}})^\mathsf{T} \in \mathbb{R}^{|\mathcal{I}_{[l]}| \times n}$. Based on this controllability, we, first, consider a simple situation in which some of the original clusters are reducible in the following sense:

Proposition 3: There exists a row-fullrank matrix $\mathsf{q}_{[l]} \in \mathbb{R}^{\left(|\mathcal{I}_{[l]}| - \delta_l\right) \times |\mathcal{I}_{[l]}|}$ such that

$$\mathsf{q}_{[l]} (e^n_{\mathcal{I}_{[l]}})^\mathsf{T} g(s) = 0 \tag{11}$$

if and only if the state of the cluster $\mathcal{I}_{[l]}$ is not clusterwise controllable. Hereafter, if (11) holds, the cluster $\mathcal{I}_{[l]}$ is said to be *reducible*.

Proof: The result immediately follows from the fact that the state of $\mathcal{I}_{[l]}$ is not clusterwise controllable if and only if $(e^n_{\mathcal{I}_{[l]}})^\mathsf{T} [B, AB, \ldots, A^{n-1} B]$ is singular. ∎

This cluster reducibility represents the uncontrollability of the state of the cluster $\mathcal{I}_{[l]}$. The following theorem characterizes the reducibility of $\mathcal{I}_{[l]}$ through the positive tri-diagonalization with respect to each input. In what follows, we define the label set $\mathbb{M} := \{1, \dots, m\}$ and denote $B = [b_1, \dots, b_m] \in \mathbb{R}^{n \times m}$.

Theorem 1: Given a dynamical network (A, B), let $\langle \mathfrak{A}_k, \mathfrak{b}_k \rangle$ be a positive tri-diagonal pair of $\langle A, b_k \rangle$ for each $k \in \mathbb{M}$ and denote its positive tri-diagonalizing matrix by H_k. Define $\mathfrak{g}_k := -\mathfrak{A}_k^{-1} \mathfrak{b}_k \in \mathbb{R}^n$ and

$$\mathsf{H}^{\mathfrak{g}}_{k[l]} := (e^n_{\mathcal{I}_{[l]}})^{\mathsf{T}} \mathsf{H}^{\mathfrak{g}}_k \in \mathbb{R}^{|\mathcal{I}_{[l]}| \times n}, \ \mathsf{H}^{\mathfrak{g}}_k := H_k \mathrm{diag}(\mathfrak{g}_k). \quad (12)$$

Then, (11) is equivalent to

$$\mathsf{q}_{[l]} \left[\mathsf{H}^{\mathfrak{g}}_{1[l]}, \dots, \mathsf{H}^{\mathfrak{g}}_{m[l]} \right] = 0. \quad (13)$$

Furthermore, take $\mathsf{p}_{[l]} \in \mathbb{R}^{\delta_l \times |\mathcal{I}_{[l]}|}$ such that $[\mathsf{p}_{[l]}^{\mathsf{T}}, \mathsf{q}_{[l]}^{\mathsf{T}}]^{\mathsf{T}} \in \mathbb{R}^{|\mathcal{I}_{[l]}| \times |\mathcal{I}_{[l]}|}$ with $\mathsf{q}_{[l]} \in \mathbb{R}^{(|\mathcal{I}_{[l]}| - \delta_l) \times |\mathcal{I}_{[l]}|}$ satisfying (13) is a unitary matrix [1]. Then, the aggregated model $(\mathsf{P}A\mathsf{P}^{\mathsf{T}}, \mathsf{P}B)$ associated with P in (4) is a stable dynamical network and satisfies

$$g(s) = \hat{\mathsf{g}}(s). \quad (14)$$

Proof: [(11)⇒(13)] Define $\mathfrak{G}_k(s) := (sI_n - \mathfrak{A}_k)^{-1} \mathfrak{b}_k$. Denote the dimension of the controllable subspace of $\langle A, b_k \rangle$ by $\bar{i}_k$, and the jth entry of $\mathfrak{G}_k$ by $\mathfrak{G}_k^j$. Since $\mathfrak{G}_k^j(s) \equiv 0$ holds for all $j \in \{\bar{i}_k + 1, \dots, n\}$

$$e^n_{1:\bar{i}_k} (e^n_{1:\bar{i}_k})^{\mathsf{T}} \mathfrak{G}_k(s) = \mathfrak{G}_k(s)$$

holds. Thus, we have

$$\begin{aligned} &\mathsf{q}_{[l]} (e^n_{\mathcal{I}_{[l]}})^{\mathsf{T}} g(s) \quad (15) \\ &= \mathsf{q}_{[l]} (e^n_{\mathcal{I}_{[l]}})^{\mathsf{T}} [H_1 \mathfrak{G}_1(s), \dots, H_m \mathfrak{G}_m(s)] \\ &= \mathsf{q}_{[l]} (e^n_{\mathcal{I}_{[l]}})^{\mathsf{T}} \times \\ &\qquad [H_1 e^n_{1:\bar{i}_1} (e^n_{1:\bar{i}_1})^{\mathsf{T}} \mathfrak{G}_1(s), \dots, H_m e^n_{1:\bar{i}_m} (e^n_{1:\bar{i}_m})^{\mathsf{T}} \mathfrak{G}_m(s)]. \end{aligned}$$

As shown in Proposition 2, the relative degree of each $\mathfrak{G}_k^j$ is j. This means that $\mathfrak{G}_k^j$ for $j \in \{1, \dots, \bar{i}_k\}$ are linearly independent each other. Therefore, the last term of (15) is equal to 0 if and only if $\mathsf{q}_{[l]} (e^n_{\mathcal{I}_{[l]}})^{\mathsf{T}} H_k e^n_{1:\bar{i}_k} = 0$ holds for all $k \in \mathbb{M}$. Hence

$$\mathsf{q}_{[l]} \mathsf{H}^{\mathfrak{g}}_{k[l]} = \mathsf{q}_{[l]} (e^n_{\mathcal{I}_{[l]}})^{\mathsf{T}} H_k e^n_{1:\bar{i}_k} (e^n_{1:\bar{i}_k})^{\mathsf{T}} \mathrm{diag}(\mathfrak{g}_k) = 0$$

follows from the fact that $\mathfrak{g}_k = e^n_{1:\bar{i}_k} (e^n_{1:\bar{i}_k})^{\mathsf{T}} \mathfrak{g}_k$.

[(13)⇒(11)] By denoting $\mathfrak{g}_k = [\mathfrak{g}_k^1, \dots, \mathfrak{g}_k^n]^{\mathsf{T}} \in \mathbb{R}^n$ and

$$\left(e_i^{|\mathcal{I}_{[l]}| - \delta_l} \right)^{\mathsf{T}} \mathsf{q}_{[l]} = \left[\mathsf{q}_{[l]}^{i,1}, \cdots, \mathsf{q}_{[l]}^{i,|\mathcal{I}_{[l]}|} \right] \in \mathbb{R}^{1 \times |\mathcal{I}_{[l]}|}$$

$$(e^n_{\mathcal{I}_{[l]}})^{\mathsf{T}} H_k = \begin{bmatrix} h_{k[l]}^{1,1} & \cdots & h_{k[l]}^{1,n} \\ \vdots & \cdots & \vdots \\ h_{k[l]}^{|\mathcal{I}_{[l]}|,1} & \cdots & h_{k[l]}^{|\mathcal{I}_{[l]}|,n} \end{bmatrix} \in \mathbb{R}^{|\mathcal{I}_{[l]}| \times n},$$

(13) is rewritten as

$$\sum_{j=1}^{|\mathcal{I}_{[l]}|} \mathsf{q}_{[l]}^{i,j} h_{k[l]}^{j,r} \mathfrak{g}_k^r = 0, \ \begin{cases} \forall i \in \{1, \dots, |\mathcal{I}_{[l]}| - \delta_l\} \\ \forall r \in \{1, \dots, n\} \end{cases} \quad (16)$$

for all $k \in \mathbb{M}$. Denoting $g(s) = [g_1(s), \dots, g_m(s)]$ and using Proposition 2 that shows $\|\mathfrak{G}_k^r\|_\infty = \mathfrak{g}_k^r$, we have

$$\begin{aligned} &\left\| \left(e_i^{|\mathcal{I}_{[l]}| - \delta_l} \right)^{\mathsf{T}} \mathsf{q}_{[l]} (e^n_{\mathcal{I}_{[l]}})^{\mathsf{T}} g_k(s) \right\|_\infty \quad (17) \\ &= \left\| \sum_{r=1}^{n} \sum_{j=1}^{|\mathcal{I}_{[l]}|} \mathsf{q}_{[l]}^{i,j} h_{k[l]}^{j,r} \mathfrak{G}_k^r(s) \right\|_\infty \le \sum_{r=1}^{n} \left| \sum_{j=1}^{|\mathcal{I}_{[l]}|} \mathsf{q}_{[l]}^{i,j} h_{k[l]}^{j,r} \mathfrak{g}_k^r \right| \end{aligned}$$

where the last term is 0 from (16).

[Poof of (14)] The stability of g follows from the fact that $\mathsf{P}A\mathsf{P}^{\mathsf{T}}$ is negative definite. Transforming the coordinate by a unitary matrix $[\mathsf{P}^{\mathsf{T}}, \overline{\mathsf{P}}^{\mathsf{T}}]^{\mathsf{T}}$, we have

$$\begin{aligned} g(s) &= \hat{\mathsf{g}}(s) + \Xi(s) \overline{\mathsf{P}} (sI_n - A)^{-1} B, \quad (18) \\ \Xi(s) &= \mathsf{P}^{\mathsf{T}} \left(sI_\Delta - \mathsf{P}A\mathsf{P}^{\mathsf{T}} \right)^{-1} \mathsf{P}A\overline{\mathsf{P}}^{\mathsf{T}} + \overline{\mathsf{P}}^{\mathsf{T}}, \end{aligned}$$

where Ξ is stable. Define $\overline{\mathsf{P}}$ by replacing $\mathsf{p}_{[l]}$ in (4) with $\mathsf{q}_{[l]}$ for each $l \in \mathbb{L}$. Then, $[\mathsf{P}^{\mathsf{T}}, \overline{\mathsf{P}}^{\mathsf{T}}]^{\mathsf{T}}$ is a unitary matrix, and $\overline{\mathsf{P}} (sI_n - A)^{-1} B = 0$ follows from (11). ■

This theorem shows that the cluster reducibility is characterized by the singularity of the set of all $\mathsf{H}^{\mathfrak{g}}_{k[l]}$. In other words, the clusterwise controllability can be determined through the basis transformation with respect to each input. This basis transformation enables us to generalize our result for single-input dynamical networks [11] to that for multi-input ones. Note, however, that (13) is generally a strict condition. That is because, it represents the uncontrollability of the state of $\mathcal{I}_{[l]}$ in terms of *all* the inputs.

B. Reducibility Relaxation and Cluster Determination

In this subsection, taking a sight on major order reduction, we relax the reducibility of $\mathcal{I}_{[l]}$ in Proposition 3 based on its equivalent condition in (13). Here, let $\overline{\delta}$ be a natural number and suppose $\overline{\delta} = \delta_1 = \cdots = \delta_L$ in (4). This means that all the original clusters are aggregated into the $\overline{\delta}$th dimensional variables. Furthermore, let $\{p_k\}_{k \in \mathbb{M}}$ be a set of row vectors where $p_1^{\mathsf{T}}, \dots, p_m^{\mathsf{T}} \in \mathbb{R}^n$ are not necessarily linearly independent but satisfy

$$\mathrm{rank}\left([p_1^{\mathsf{T}}, \dots, p_m^{\mathsf{T}}] \right) = \overline{\delta}.$$

For this $\{p_k\}_{k \in \mathbb{M}}$, if we take $\mathsf{p}_{[l]} \in \mathbb{R}^{\overline{\delta} \times |\mathcal{I}_{[l]}|}$ such that [2]

$$\mathrm{span}\left(\mathsf{p}_{[l]}^{\mathsf{T}} \right) = \mathrm{span}\left((e^n_{\mathcal{I}_{[l]}})^{\mathsf{T}} [p_1^{\mathsf{T}}, \dots, p_m^{\mathsf{T}}] \right), \quad (19)$$

then

$$\mathrm{span}\left([p_1^{\mathsf{T}}, \dots, p_m^{\mathsf{T}}] \right) \subseteq \mathrm{span}\left(\mathsf{P}^{\mathsf{T}} \right) \quad (20)$$

holds. This implies that $\mathrm{span}(\mathsf{P}^{\mathsf{T}})$, which coincides with the controllable subspace of the aggregated model $(\mathsf{P}A\mathsf{P}^{\mathsf{T}}, \mathsf{P}B)$, includes the space spanned by the set of the row vectors

[1] We allow empty $\mathsf{q}_{[l]}$.

[2] If $\overline{\delta} \ge |\mathcal{I}_{[l]}|$, we take $\mathsf{p}_{[l]} = I_{|\mathcal{I}_{[l]}|}$.

$\{p_k\}_{k\in\mathbb{M}}$. Accordingly to this fact, we can exploit the freedom of $\{p_k\}_{k\in\mathbb{M}}$ to impose some specific properties on the aggregated model, such as the preservation of the steady-state characteristic and the positivity (see Definition 1) of dynamical networks. We give the following definition:

Definition 5: Given a dynamical network (A, B), let $\langle \mathfrak{A}_k, \mathfrak{b}_k \rangle$ be a positive tri-diagonal pair of $\langle A, b_k \rangle$ for each $k \in \mathbb{M}$ and denote its positive tri-diagonalizing matrix by H_k. Furthermore, for a given set $\{p_k\}_{k\in\mathbb{M}}$ of row vectors $p_k \in \mathbb{R}^{1\times n}$, which are not necessarily linearly independent, denote

$$p_k e^n_{\mathcal{I}_{[l]}} = \left[p^1_{k[l]}, \ldots, p^{|\mathcal{I}_{[l]}|}_{k[l]}\right] \in \mathbb{R}^{1\times|\mathcal{I}_{[l]}|}.$$

The cluster $\mathcal{I}_{[l]}$ for $l \in \mathbb{L}$ is said to be *θ-weakly reducible* with respect to $\{p_k\}_{k\in\mathbb{M}}$ if $p^1_{k[l]} \neq 0$ and $\mathsf{H}^{\mathfrak{g}}_{k[l]}$ in (12) satisfies

$$\left\| \mathsf{h}^{j\mathfrak{g}}_{k[l]} - \frac{p^j_{k[l]}}{p^1_{k[l]}} \mathsf{h}^{1\mathfrak{g}}_{k[l]} \right\| \leq \theta,\ \theta \in \mathbb{R}_+ \tag{21}$$

for all $k \in \mathbb{M}$ and $j \in \{1, \ldots, |\mathcal{I}_{[l]}|\}$, where $\mathsf{h}^{j\mathfrak{g}}_{k[l]} \in \mathbb{R}^{1\times n}$ denotes the jth row vectors of $\mathsf{H}^{\mathfrak{g}}_{k[l]}$.

In this definition, the constant θ prescribes the degree of the linear dependence between $\mathsf{h}^{j\mathfrak{g}}_{k[l]}$ and $\mathsf{h}^{1\mathfrak{g}}_{k[l]}$ scaled by $p^j_{k[l]}/p^1_{k[l]}$. Namely, θ represents the degree of the reducibility of $\mathcal{I}_{[l]}$ under the weighting $\mathsf{p}_{[l]}$ in (19). In addition, the weak cluster reducibility suitable for multi-input systems is defined by introducing $\{p_k\}_{k\in\mathbb{M}}$.

In what follows, we construct a cluster set $\{\mathcal{I}_{[l]}\}_{l\in\mathbb{L}}$ such that all the clusters are θ-weakly reducible. Here, we exploit the freedom of $\{p_k\}_{k\in\mathbb{M}}$ to achieve $g(0) = \hat{\mathsf{g}}(0)$. The aggregation of the θ-weakly reducible cluster set yields the aggregated model having the properties as shown in the following theorem:

Theorem 2: Given a dynamical network (A, B), let $\langle \mathfrak{A}_k, \mathfrak{b}_k \rangle$ be a positive tri-diagonal pair of $\langle A, b_k \rangle$ for each $k \in \mathbb{M}$ and denote its positive tri-diagonalizing matrix by H_k. Take $p_k = \mathfrak{g}^{\mathsf{T}}_k H_k$ for all $k \in \mathbb{M}$. For each θ-weakly reducible cluster $\mathcal{I}_{[l]}$ with respect to $\{p_k\}_{k\in\mathbb{M}}$, define $\mathsf{p}_{[l]} \in \mathbb{R}^{m\times|\mathcal{I}_{[l]}|}$ such that

$$\begin{cases} \mathrm{span}(\mathsf{p}^{\mathsf{T}}_{[l]}) = \mathrm{span}\left((e^n_{\mathcal{I}_{[l]}})^{\mathsf{T}} \left[p^{\mathsf{T}}_1, \ldots, p^{\mathsf{T}}_m\right]\right), & \text{if } m < |\mathcal{I}_{[l]}| \\ \mathsf{p}_{[l]} = I_{|\mathcal{I}_{[l]}|}, & \text{otherwise.} \end{cases} \tag{22}$$

Then, the aggregated model $(\mathsf{P}A\mathsf{P}^{\mathsf{T}}, \mathsf{P}B)$ associated with P in (4) is a stable dynamical network, and satisfies $g(0) = \hat{\mathsf{g}}(0)$ and

$$\|g(s) - \hat{\mathsf{g}}(s)\|_\infty \leq \alpha\theta \tag{23}$$

for a positive constant α.

Proof: The stability of $\hat{\mathsf{g}}$ follows from the fact that $\mathsf{P}A\mathsf{P}^{\mathsf{T}}$ is negative definite. We prove (23) based on (18). Note that $\|\mathsf{P}\| = \|\overline{\mathsf{P}}\| = 1$ and

$$\left\|\left(sI_\Delta - \mathsf{P}A\mathsf{P}^{\mathsf{T}}\right)^{-1}\right\|_\infty = \left\|\left(\mathsf{P}A\mathsf{P}^{\mathsf{T}}\right)^{-1}\right\| \leq \left\|A^{-1}\right\|,$$

which follows from the Cauchy interlacing theorem [5]. Therefore, we have the bound of the norm of Ξ in (18) as

$$\|\Xi(s)\|_\infty \leq \|A\| \left\|A^{-1}\right\| + 1$$

which does not depend on P. Thus, what remains to be shown is

$$\left\|\overline{\mathsf{P}}\left(sI_n - A\right)^{-1} B\right\|_\infty \leq c\theta \tag{24}$$

for a positive constant c. We rewrite the matrix $\mathsf{H}^{\mathfrak{g}}_{k[l]}$ in (12) as

$$\mathsf{H}^{\mathfrak{g}}_{k[l]} = \frac{\mathsf{h}^{1\mathfrak{g}}_{k[l]}}{p^1_{k[l]}} \left(p_k e^n_{\mathcal{I}_{[l]}}\right)^{\mathsf{T}} + \left[0, \eta^{\mathsf{T}}_{k,2}, \ldots, \eta^{\mathsf{T}}_{k,|\mathcal{I}_{[l]}|}\right]^{\mathsf{T}}$$

where

$$\eta_{k,j} := \mathsf{h}^{j\mathfrak{g}}_{k[l]} - \frac{p^j_{k[l]}}{p^1_{k[l]}} \mathsf{h}^{1\mathfrak{g}}_{k[l]} \in \mathbb{R}^{1\times n}.$$

There exists $\mathsf{q}_{[l]}$ such that $[\mathsf{p}^{\mathsf{T}}_{[l]}, \mathsf{q}^{\mathsf{T}}_{[l]}]^{\mathsf{T}}$ is a unitary matrix [3]. Thus, we have

$$\mathsf{q}_{[l]} \mathsf{H}^{\mathfrak{g}}_{k[l]} = \mathsf{q}_{[l]} \left[0, \eta^{\mathsf{T}}_{k,2}, \ldots, \eta^{\mathsf{T}}_{k,|\mathcal{I}_{[l]}|}\right]^{\mathsf{T}}$$

from the relation (22). Note that $\|\mathsf{q}_{[l]}\| = 1$. Moreover, the definition of the θ-weak reducibility implies $\|\eta_{k,j}\| \leq \theta$ for all $k \in \mathbb{M}$ and $j \in \{2, \ldots, |\mathcal{I}_{[l]}|\}$. Hence, (24) follows from the same argument in the proof of Theorem 1. Finally, from the inclusion (20) with $p_k = \mathfrak{g}^{\mathsf{T}}_k H_k = (-A^{-1}b_k)^{\mathsf{T}}$, we have $\overline{\mathsf{P}}A^{-1}B = 0$ in (18). Hence, $g(0) = \hat{\mathsf{g}}(0)$ follows. ■

Theorem 2 indicates that by taking the aggregation matrix with (22), we can construct the aggregated model such that the discrepancy between the transfer functions $\hat{\mathsf{g}}$ and g is *linearly* bounded by θ and their DC-gain is identical. In addition, $\mathsf{p}_{[l]}$ satisfying (22) is constructed by using, e.g., the Gram-Schmidt orthogonalization of $(e^n_{\mathcal{I}_{[l]}})^{\mathsf{T}}[p^{\mathsf{T}}_1, \ldots, p^{\mathsf{T}}_m]$.

For the multi-input cases (i.e., $m \geq 2$), $\mathsf{p}_{[l]}$ in Theorem 2 has negative entries even if (A, B) is a *positive* dynamical network. This means that the aggregated model does not preserve the positivity of the original system. The following theorem shows that if all the clusters are aggregated into scaler variables (i.e., if $\delta_1 = \cdots = \delta_L = 1$), the aggregated model preserves the positivity:

Theorem 3: Given a positive dynamical network (A, B), take $p_1 = \cdots = p_m = \mathfrak{g}^{\mathsf{T}}_\mu H_\mu,\quad \mu \in \mathbb{M}$ under the same notation as in Theorem 2. Furthermore, for each θ-weakly reducible cluster $\mathcal{I}_{[l]}$ with respect to $\{p_k\}_{k\in\mathbb{M}}$, define

$$\mathsf{p}_{[l]} = \frac{p_{\mu[l]}}{\|p_{\mu[l]}\|} \in \mathbb{R}^{1\times|\mathcal{I}_{[l]}|},\ p_{\mu[l]} := \mathfrak{g}^{\mathsf{T}}_\mu H_\mu e^n_{\mathcal{I}_{[l]}}. \tag{25}$$

Then, the aggregated model $(\mathsf{P}A\mathsf{P}^{\mathsf{T}}, \mathsf{P}B)$ associated with P in (4) is a stable positive dynamical network, and satisfies (23) for a positive constant α. In addition $g_\mu(0) = \hat{\mathsf{g}}_\mu(0)$ holds, where g_μ and $\hat{\mathsf{g}}_\mu$ are the μth column vectors of g and $\hat{\mathsf{g}}$, respectively.

Proof: The result of the positivity preservation follows from the similar argument in the proof of Corollary 2 in [13]. Furthermore, since $\overline{\mathsf{P}}A^{-1}b_\mu = 0$ holds for (18), $g_\mu(0) = \hat{\mathsf{g}}_\mu(0)$ follows. ■

[3] We allow empty $\mathsf{q}_{[l]}$.

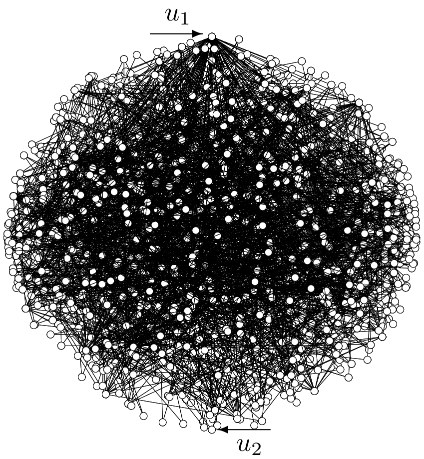

Fig. 3. Dynamical System Evolving on the Holme-Kim Model (1000 Nodes).

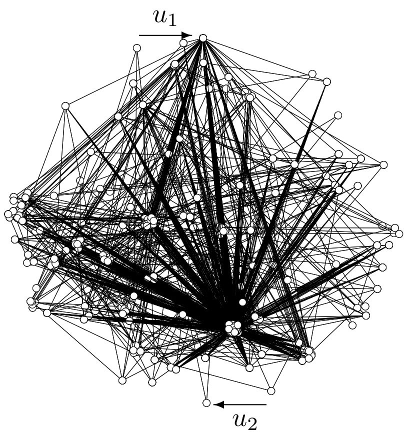

Fig. 4. Clusterized Network of the Holme-Kim Model (103 Clusters).

C. Numerical Example

In this subsection, we show the efficiency of the proposed clustering method through a numerical example. We deal with a linear system evolving on the complex network of the Holme-Kim model in Fig. 3, which is well-known as an extension of the Barabasi-Albert model satisfying the scale-free and small-world property while having the high cluster coefficient [1], [3]. This graph has 1000 nodes and 2000 edges.

Let be given the bidirectional network (A, b) as follows: For $A \in \mathbb{R}^{3000\times 3000}$ in (2), the diffusion terms $a_{i,j}$ are randomly chosen from $(0, 1]$ if node i and j of $i \neq j$ are connected, otherwise 0, and the reaction terms are taken as $r_1 = 1$ and $r_i = 0$ for all $i \in \{2, \ldots, 3000\}$. Moreover, we take $B = [I_2, 0] \in \mathbb{R}^{1000\times 2}$, i.e., the first and second nodes are the input nodes.

By Theorem 2, we construct an aggregated model preserving the DC-gain. Taking the coarseness parameter as $\theta = 0.5$, we obtain the aggregated model with the 103rd order, whose interconnection topology is shown in Fig. 4. Comparing this figure with Fig. 3, we can see that relatively far nodes from the inputs u_1 and u_2 are remarkably clusterized. Since the relative error $\|g - \hat{\mathrm{g}}\|_{\mathcal{H}_\infty}/\|g\|_{\mathcal{H}_\infty}$ is 0.0032, the original system is well approximated by the aggregated model.

IV. Conclusion

In this paper, we have proposed a model reduction method for a linear system evolving on large-scale complex networks, called dynamical networks. The main contribution of this paper is generalization of the reduction scheme for single-input dynamical networks [11] to that for multi-input ones. To this end, we have introduced a basis transformation of the state-space that is implemented as matrix tridiagonalization with respect to each input. Using this basis transformation, we construct a set of clusters (i.e., disjoint subsets of state variables) based on a notion of clusterwise controllability that characterizes local controllability of the state-space of the dynamical networks. Aggregating the cluster set under suitable weighting, we obtain a reduced model that preserves interconnection topology of the original system as well as some specific properties, such as stability, steady-state characteristic and system positivity. Furthermore, we have derived an H_∞-error bound of the state discrepancy caused by the aggregation.

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