Hierarchical Decentralized Observers for Networked Linear Systems

Takayuki Ishizaki[1,2], Masakazu Koike[1,2], Tomonori Sadamoto[1], Jun-ichi Imura[1,2]

***Abstract*— In this paper, we propose a design method of hierarchical decentralized observers for networked linear systems. In this method, based on suitable state-space expansion of the network systems, we, first, find a high-dimensional dynamical compensator that can achieve ideal performance for decentralized state estimation. Next, fully utilizing model reduction techniques, we extract a subspace that is essentially relevant to the decentralized state estimation, from the high-dimensional state-space of the dynamical compensator. This procedure successfully produces a lower-dimensional compensator that guarantees not only the stability of the estimation error but also desirable estimation performance with respect to tracking the system behavior for external input signals. Finally, the efficiency of the proposed method is shown through an illustrative example of power networks.**

I. Introduction

With the recent development of communication and computation technology, the architecture of systems in engineering have tended to become more complex and larger in scale. Typically, such large-scale complex systems are spatially distributed and networked. In view of this, it is crucial to build a framework for designing distributed/decentralized control systems having good compatibility with the spatial distribution of networked systems [1], [2].

Major examples of networked systems include power systems. Currently, in the research area of power systems, renewable energy, such as photovoltaic power generation and wind-power generation, is expected to be a key to the solution of environment and energy problems [3]. However, a systematic use of renewable energy is not necessarily straightforward. This is because the amount of power generation varies regionally, depending on, e.g., local climate condition. To overcome this difficulty, it is reasonable to control the power generation and transmission in a distributed fashion. It is well known that distributed/decentralized control has an advantage to substantially reduce the costs of communication (computation) among networked systems. However, as discussed in literature, designing control systems with imposing a specific structure is not necessarily easy [4], [5], [6].

As one effective approach to distributed control, a method based on Inclusion Principle has been developed in [5], [6], and some illustrative application to the distributed control of vehicles is given in [7]. The design procedure of this method can be implemented in a systematic fashion, yet the resultant control system is often conservative from the viewpoint of control performance. This is due to the fact that the interaction among subsystems is not dealt with quantitatively, and a set of decentralized controllers for expanded systems must be designed so as to be exactly contractable.

Towards the development of a systematic framework for distributed/decentralized control, we propose a method to design a novel type of structured observers for networked systems, called a *hierarchical decentralized observer*. In this structured observer, a dynamical compensator is employed in conjunction with a set of decentralized observers. The architecture of observers was introduced by the authors in [8], where a networked system composed of identical subsystems is dealt with.

To design the dynamical compensator systematically, we decouple the state-space of networked systems into state-spaces associated with individual subsystems and the interaction among subsystems. It should be remarked that the decoupling of the state-space cannot be realized by usual coordinate transformations. Instead, we utilize state-space expansion similar to the method based on Inclusion Principle. More specifically, we introduce some redundancy into the state-space of networked systems so as to realize a structure preferable to a distinctive use of different kinds of sensor signals. This redundancy provides some flexibility for deriving an ideal dynamical compensator that can achieve desirable performance of the state estimation, though it inevitably imposes high-dimensionality on the compensator.

To design a more practical, i.e., lower-dimensional, dynamical compensator, we finely approximate the ideal compensator obtained above. We show that significant order reduction can be achieved by suitably relaxing the state estimation problem from a viewpoint of the $\mathcal{H}_\infty$-control theory. This design approach to hierarchical decentralized observers is fairly novel in the sense that:

- Based on the state-space expansion of systems, we find a high-dimensional compensator that achieves ideal performance of the state estimation.
- By fully utilizing a model reduction technique, we extract a subspace essential for the state estimation from the state-space of the high-dimensional compensator.

The remainder of this paper is structured as follows: In Section II, we formulate a design problem of hierarchical decentralized observers, and show, based on state-space expansion, that an ideal compensator can be designed independently of designing a set of decentralized observers for local subsystems. Next, in Section III, we relax the design problem of dynamical compensators so as to allow an

[1]Department of Mechanical and Environmental Informatics, Graduate School of Information Science and Engineering, Tokyo Institute of Technology; 2-12-1, Meguro, Tokyo, Japan:
{ishizaki, imura}@mei.titech.ac.jp
{koike, sadamoto}@cyb.mei.titech.ac.jp
[2]CREST, Japan Science and Technology Agency; 4-1-8, Honcho, Kawaguchi, Saitama, 332-0012, Japan.

estimation error in a reasonable sense, and then, we show that the relaxed problem is equivalently rephrased as a frequency-weighted model reduction problem. In addition. we propose an implementation procedure for the model reduction of dynamical compensators. In Section IV, we show the efficiency of the proposed hierarchical decentralized observer by an illustrative example of power networks. Finally, concluding remarks are provided in Section V.

Notation: We denote the set of real numbers by $\mathbb{R}$, the set of complex numbers by $\mathbb{C}$, the n-dimensional identity matrix by I_n, and the image of a matrix M by $\mathrm{im}(M)$. Furthermore, for $\mathbb{N} = \{1, \ldots, N\}$, we denote the block-diagonal matrix having matrices $M_1, \ldots, M_N$ on its diagonal blocks by

$$\operatorname*{diag}_{i\in\mathbb{N}}(M_i) = \mathrm{diag}(M_1, \ldots, M_N).$$

In particular, if not confusing, we omit the subscript $i \in \mathbb{N}$. The $\mathcal{L}_2$-norm of a vector-valued square-integrable function f is defined by

$$\|f(t)\|_{\mathcal{L}_2} := \sqrt{\int_0^\infty \|f(t)\|^2 dt}.$$

Finally, the $\mathcal{H}_\infty$-norm of a stable transfer matrix G is defined by

$$\|G(s)\|_{\mathcal{H}_\infty} := \sup_{\omega\in\mathbb{R}} \|G(j\omega)\|.$$

II. Hierarchical Decentralized Observer

A. Problem Formulation

In this paper, we deal with networked linear systems composed of N subsystems. The dynamics of the ith subsystem Σ_i is described by

$$\Sigma_i : \begin{cases} \dot{x}_i = A_{i,i}x_i + \sum_{j\neq i}^N A_{i,j}x_j + B_i u \\ y_i = C_i x_i \end{cases} \tag{1}$$

where $A_{i,j} \in \mathbb{R}^{n_i\times n_j}$, $B_i \in \mathbb{R}^{n_i\times m}$, and $C_i \in \mathbb{R}^{p_i\times n_i}$. For this networked system, we consider a set of decentralized observers estimating the state x_i of Σ_i by using the sensor signal y_i. The decentralized observer associated with the ith subsystem is described by

$$O_i : \begin{cases} \dot{\hat{x}}_i = A_{i,i}\hat{x}_i + B_i u + h_i(y_i - \hat{y}_i) + z_i \\ \hat{y}_i = C_i\hat{x}_i \end{cases} \tag{2}$$

where $h_i \in \mathbb{R}^{n_i\times p_i}$ denotes a feedback gain for $y_i \in \mathbb{R}^{p_i}$, and $z_i \in \mathbb{R}^{n_i}$ denotes an *additional* input for compensation. In the rest of this paper, we assume that each $A_{i,i} - h_iC_i$ is stable and $\hat{x}_i(0) = 0$ for all $i \in \mathbb{N} := \{1, \ldots, N\}$.

We design a dynamical compensator that produces a desirable z_i by using an additional sensor signal, denoted by $r \in \mathbb{R}^{p_r}$. For the state

$$x := [x_1^\mathsf{T}, \ldots, x_N^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^n, \quad n := n_1 + \cdots + n_N,$$

the dynamics of the whole networked system is represented by

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = \mathrm{diag}(C_i)x \\ r = Sx \end{cases} \tag{3}$$

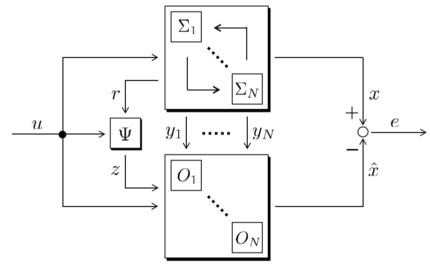

Fig. 1. Structure of Hierarchical Decentralized Observer.

where $S \in \mathbb{R}^{p_r\times n}$ and

$$A := \begin{bmatrix} A_{1,1} & \cdots & A_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1} & \cdots & A_{N,N} \end{bmatrix}, \quad B := \begin{bmatrix} B_1 \\ \vdots \\ B_N \end{bmatrix}.$$

We describe an ν-dimensional dynamical compensator that produces $z = [z_1^\mathsf{T}, \ldots, z_N^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^n$ from r by

$$\Psi : \begin{cases} \dot{\psi} = \mathbf{E}\psi + \mathbf{F}u + \mathbf{H}r \\ z = \mathbf{G}\psi \end{cases} \tag{4}$$

where the design parameters are denoted by $\mathbf{E} \in \mathbb{R}^{\nu\times\nu}$, $\mathbf{F} \in \mathbb{R}^{\nu\times m}$, $\mathbf{G} \in \mathbb{R}^{n\times\nu}$, and $\mathbf{H} \in \mathbb{R}^{\nu\times p_r}$. In what follows, we denote the initial state of Σ by $x(0) = x_0$ and assume that the pair (A, S) is detectable and $\psi(0) = 0$.

Finally, we denote the estimation error via $\{O_i\}_{i\in\mathbb{N}}$ by

$$e(t; u, x_0) := x - \hat{x} \tag{5}$$

where $\hat{x} := [\hat{x}_1^\mathsf{T}, \ldots, \hat{x}_N^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^n$. In this notation, we formulate a design problem of Ψ as follows:

Problem 1: Let Σ in (3) be given with $\{O_i\}_{i\in\mathbb{N}}$ in (2). Find Ψ in (4) such that

$$\lim_{t\to\infty} e(t; 0, x_0) = 0 \tag{6}$$

holds for any x_0, and

$$e(t; u, 0) = 0, \quad t \geq 0 \tag{7}$$

holds for any u, where e is defined as in (5).

Fig. 1 depicts the communication structure among the observers and the networked system. In this paper, we call this structured observer, consisting of $\{O_i\}_{i\in\mathbb{N}}$ and Ψ, a *hierarchical decentralized observer*.

In Problem 1, we give (6) and (7) as a design specification of Ψ; namely it is required that e is stable, and the effect of the external input u is exactly eliminated by the dynamical behavior of the hierarchical decentralized observer. The exact elimination of the effect of u is actually standard for state estimation problems [9]. For instance, since the Luenberger-type observer *copies* the input term of systems, the effect of input signals does not appear in the the estimation error dynamics, i.e., (7) is satisfied. It should be noted that it is important to explicitly take into account the input signal effect for estimation errors, especially in the design of low-dimensional observers; see Section III for details.

B. Dynamical Compensator Design Based on State-Space Expansion

Problem 1 clearly contrasts with standard state estimation problems in the sense that the different sensor signals y and r are used in $\{O_i\}_{i\in\mathbb{N}}$ and Ψ, respectively. To use the different sensor signals in this distinctive manner, we aim to transform the realization of Σ into a desirable one. The following lemma shows that such a system transformation can be realized based on state-space expansion:

Lemma 1: Let Σ in (3) be given, and define

$$\tilde{\Sigma}: \left\{ \begin{array}{rcl} \dot{\tilde{x}} &=& \tilde{A}\tilde{x} + \tilde{B}u \\ \tilde{y} &=& \tilde{C}\tilde{x}, \end{array} \right. \tag{8}$$

where

$$\begin{array}{l} \tilde{A} := \begin{bmatrix} \mathrm{diag}(A_{i,i}) & \Gamma \\ 0 & A \end{bmatrix}, \quad \tilde{B} := \begin{bmatrix} B \\ B \end{bmatrix} \\ \tilde{C} := \begin{bmatrix} \mathrm{diag}(C_i) & 0 \\ 0 & S \end{bmatrix}, \end{array} \tag{9}$$

with

$$\Gamma := A - \mathrm{diag}(A_{i,i}). \tag{10}$$

If $\tilde{x}(0) = [x_0^{\mathsf{T}}, x_0^{\mathsf{T}}]^{\mathsf{T}} \in \mathbb{R}^{2n}$, then it follows that

$$\tilde{x}(t) = \begin{bmatrix} x(t) \\ x(t) \end{bmatrix}, \quad \tilde{y}(t) = \begin{bmatrix} y(t) \\ r(t) \end{bmatrix}, \quad t \geq 0 \tag{11}$$

for all u.

Lemma 1 gives a desirable realization of Σ to use y and r distinctively. Note that some redundancy is introduced in $\tilde{\Sigma}$ owing to the state-space expansion. Utilizing this redundancy, we derive the following theorem that gives a solution to Problem 1:

Theorem 1: Let Σ in (3) be given with $\{O_i\}_{i\in\mathbb{N}}$ in (2). Furthermore, let $H \in \mathbb{R}^{n\times p_r}$ such that $A - HS$ is stable. If

$$\mathbf{E} = A - HS, \quad \mathbf{F} = B, \quad \mathbf{H} = H, \quad \mathbf{G} = \Gamma \tag{12}$$

where Γ is defined as in (10), then Ψ in (4) satisfies (6) and (7) for any x_0 and u.

Proof: Consider $\tilde{\Sigma}$ in (8). From Lemma 1, it follows that $y = \mathrm{diag}(C_i)\tilde{x}_1$ and $r = S\tilde{x}_2$. Thus, we obtain

$$\begin{array}{rcl} \dot{\hat{x}} &=& \mathrm{diag}(A_{i,i} - h_iC_i)\hat{x} + Bu + \mathrm{diag}(h_iC_i)\tilde{x}_1 + \Gamma\psi \\ \dot{\psi} &=& (A - HS)\psi + Bu + HS\tilde{x}_2 \end{array}$$

where $\tilde{x} = [\tilde{x}_1^{\mathsf{T}}, \tilde{x}_2^{\mathsf{T}}]^{\mathsf{T}}$. Let $e_\psi := \tilde{x}_2 - \psi$. Noting that $e = \tilde{x}_1 - \hat{x}$, we obtain

$$\begin{bmatrix} \dot{e} \\ \dot{e}_\psi \end{bmatrix} = \begin{bmatrix} \mathrm{diag}(A_{i,i} - h_iC_i) & \Gamma \\ 0 & A - HS \end{bmatrix} \begin{bmatrix} e \\ e_\psi \end{bmatrix}.$$

Therefore, the stability of $A - HS$ ensures that (6) holds for any x_0. Furthermore, it is clear that (7) holds for any u if $\tilde{x}_1(0) = \tilde{x}_2(0) = x_0 = 0$. Hence, the claim follows. ■

Theorem 1 shows that Ψ given by (12), which can be obtained independently of designing $\{O_i\}_{i\in\mathbb{N}}$, satisfies the design specification given in Problem 1. Note, however, that this Ψ can be regarded as a centralized observer that uses the additional sensor signal r, since it is n-dimensional. Actually, as shown in the proof of Theorem 1, the state ψ of Ψ also converges to the state x of Σ. In this sense, this dynamical compensator is redundant.

To eliminate the redundancy of Ψ, we employ the following idea: From the fact that $z = \Gamma\psi$, we notice that Ψ in Theorem 1 is an observer to estimate Γx, which can be regarded as *interference* among Σ_i in (1). This fact indicates that Ψ does not need to estimate all the states of Σ, but it needs to estimate Γx appropriately. From this, we can expect to obtain a lower-dimensional compensator by extracting a subspace relevant to Γx from the high-dimensional state-space of Ψ in Theorem 1.

III. Lower-Dimensional Compensator Design Via Model Reduction

A. Translation to Frequency-Weighted Model Reduction Problem

In Problem 1, we require that (7) exactly holds for any u, as a design specification. Even though this requirement is standard for state estimation problems, the resultant observer is necessary to have a dimension comparable with a system to be observed. In view of this, we relax the design problem of the dynamical compensator as follows:

Problem 2: Let Σ in (3) be given with $\{O_i\}_{i\in\mathbb{N}}$ in (2). Given a constant $\epsilon \geq 0$, find Ψ in (4) such that (6) holds for any x_0, and

$$\sup_{u\neq 0} \frac{\|e(t;u,0)\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}} \leq \epsilon \tag{13}$$

holds, where e is defined as in (5).

In Problem 2, we relax Problem 1 so that some estimation error with respect to input signals is allowed. As shown in the following theorem, this relaxed problem can be equivalently rephrased as a model reduction problem. This theorem is the main result of this paper.

Theorem 2: Let Σ in (3) be given with $\{O_i\}_{i\in\mathbb{N}}$ in (2). If $\mathbf{E} \in \mathbb{R}^{\nu\times\nu}$ is stable, and there exists $w \in \mathbb{C}^\nu$ such that

$$\begin{bmatrix} \mathrm{diag}(A_{i,i}) & \mathbf{G} \\ \mathbf{H}S & \mathbf{E} \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \lambda \begin{bmatrix} v \\ w \end{bmatrix} \tag{14}$$

for any $(\lambda, v) \in \mathbb{C} \times \mathbb{C}^n$ satisfying

$$Av = \lambda v, \quad \mathrm{Re}(\lambda) \geq 0, \tag{15}$$

then Ψ in (4) satisfies (6) for any x_0. Furthermore, with $H \in \mathbb{R}^{n\times p_r}$ such that $A - HS$ is stable, define

$$\begin{array}{rcl} \overline{\Psi}_{\mathrm{d}}(s) &=& \Gamma(sI_n - A + HS)^{-1}[B\ H] \\ \overline{\Psi}(s) &=& \mathbf{G}(sI_\nu - \mathbf{E})^{-1}[\mathbf{F}\ \mathbf{H}], \end{array} \tag{16}$$

where Γ is defined as in (10). Then, Ψ satisfies (13) if and only if

$$\|\Delta(s)\|_{\mathcal{H}_\infty} \leq \epsilon \tag{17}$$

holds for

$$\Delta(s) := W(s)\{\overline{\Psi}_{\mathrm{d}}(s) - \overline{\Psi}(s)\}V(s) \tag{18}$$

where

$$W(s) = \operatorname{diag}\left((sI_{n_i} - A_{i,i} + h_i C_i)^{-1}\right)$$
$$V(s) = \begin{bmatrix} I_m \\ S(sI_n - A)^{-1}B \end{bmatrix}. \tag{19}$$

Proof: If there exists $w \in \mathbb{C}^\nu$ such that (14), it follows that $\mathcal{A}\mathcal{V} = \lambda\mathcal{V}$ for $\mathcal{V} = [v^\mathsf{T}, w, v^\mathsf{T}]^\mathsf{T}$. Hence, we obtain $\mathcal{C}\mathcal{V} = 0$ proving (6). Next, we prove the equivalence between (13) and (17). Let Ψ_d denote Ψ given by (12). Furthermore, when using Ψ_d, the estimation of x is denoted by $\hat{x}_\mathrm{d}$, and the estimation error is denoted by $e_\mathrm{d}(t; u, x_0) := x - \hat{x}_\mathrm{d}$. Then, it follows from Theorem 1 that $e_\mathrm{d}(t; u, 0) = 0$ for any u. Note that $\hat{x}_\mathrm{d} - \hat{x} = e(t; u, 0) - e_\mathrm{d}(t; u, 0) = e(t; u, 0)$ holds if $x_0 = 0$. Thus, to prove the equivalence between (13) and (17), it suffices to show that Δ in (18) is the transfer function from u to $\hat{x}_\mathrm{d} - \hat{x}$. We can verify that the transfer function from u to $\hat{x}$ is represented as

$$W(s)\{\overline{\Psi}(s)V(s) + Z(s)\}$$

where $Z(s) := \operatorname{diag}(h_i C_i)(sI_n - A)^{-1}B + B$. Clearly, by the definition of Ψ_d, the the transfer function from u to $\hat{x}_\mathrm{d}$ is given by $W(s)\{\overline{\Psi}_\mathrm{d}(s)V(s) + Z(s)\}$. Hence, Δ is the transfer function from u to $e = \hat{x}_\mathrm{d} - \hat{x}$. ■

In (16), $\overline{\Psi}$ is composed of the design parameter of the dynamical compensator while $\overline{\Psi}_\mathrm{d}$ is composed of the system matrices of the *ideal* one in Theorem 1. Noting that (13) is equivalent to (17), we see that Problem 2 is equivalently converted to a frequency-weighted model reduction problem for the n-dimensional compensator in Theorem 1.

B. Implementation of Model Reduction

Clearly, if Σ is stable, the input weight V is stable. Therefore, a lower-dimensional compensator that satisfies (17) can be straightforwardly obtained by frequency-weighted model reduction methods found in literature [10]. On the other hand, if Σ is not stable, V is not stable as explained above. Due to this, we cannot directly apply the existing methods. Additionally, the condition in (14), which characterizes the stability of e in (5), is not satisfied in general. In this sense, it is not straightforward to solve the frequency-weighted model reduction problem, if Σ is not stable.

In view of this, instead of directly solving the frequency-weighted model reduction problem, we propose a method to achieve that

$$\|\overline{\Psi}(s) - \overline{\Psi}_\mathrm{d}(s)\|_{\mathcal{H}_\infty}, \tag{20}$$

i.e., the approximation error of the ideal compensator, is sufficiently small, while guaranteeing the condition in (14). In what follows, we parametrize Ψ in (4) by $P \in \mathbb{R}^{\nu \times n}$ as

$$\mathbf{E} = P(A - HS)P^\mathsf{T}, \quad \mathbf{F} = PB, \quad \mathbf{H} = PH$$
$$\mathbf{G} = \Gamma P^\mathsf{T}, \tag{21}$$

where Γ is defined as in (10).

Let us consider a balanced realization of $\overline{\Psi}_\mathrm{d}$ in (16). Namely, letting

$$f(A, B) := \int_0^\infty e^{At}B(e^{At}B)^\mathsf{T}dt,$$

we consider the case where the controllability and observability gramians

$$\mathcal{P} := f(A - HS, [B\ H]), \quad \mathcal{Q} := f((A - HS)^\mathsf{T}, \Gamma^\mathsf{T}) \tag{22}$$

are identical. We denote the eigenvalues of $\mathcal{P} = \mathcal{Q}$ by $\sigma_1, \ldots, \sigma_n$ assuming $\sigma_i \geq \sigma_{i+1}$ without loss of generality. In this notation, if we take $P = U \in \mathbb{R}^{k \times n}$ such that

$$U\mathcal{P}U^\mathsf{T} = U\mathcal{Q}U^\mathsf{T} = \operatorname{diag}(\sigma_1, \ldots, \sigma_k), \tag{23}$$

then $\overline{\Psi}$ in (16) coincides with the k-dimensional approximant of $\overline{\Psi}_\mathrm{d}$ obtained by the standard balanced truncation, where the approximation error in (20) is bounded by

$$2\sum_{i \in \mathcal{I}_k} \sigma_i, \quad \mathcal{I}_k = \{i \in \{k+1, \ldots, n\} : \sigma_i \neq \sigma_{i+1}\}. \tag{24}$$

If the value in (24) is small, the approximation error in (20) is also sufficiently small. However, it should be noted that this standard approximation does not guarantee the stability of e because the condition in (14) is not satisfied in general. To ensure its stability, we give a generalization of the balanced truncation as follows:

Lemma 2: Let Σ in (3) be given with $\{O_i\}_{i \in \mathbb{N}}$ in (2). Furthermore, let $\overline{\Psi}_\mathrm{d}$ in (16), and assume that $A - HS$ is stable, $\mathcal{P} = \mathcal{Q}$ holds for (22), and the pair

$$\left(\mathcal{S}(A - HS), [B\ H\ \Gamma^\mathsf{T}]\right) \tag{25}$$

is stabilizable, where $\mathcal{S}(X) := X + X^\mathsf{T}$. If $P \in \mathbb{R}^{\nu \times n}$ satisfies

$$\operatorname{im}(v) \subseteq \operatorname{im}(P^\mathsf{T}), \quad PP^\mathsf{T} = I_\nu \tag{26}$$

for any $v \in \mathbb{C}^n$ such that (15), then Ψ in (4) given by (21) satisfies (6) for any x_0.

As shown in Lemma 2, the stability of e in (5) is ensured by any orthogonal projection satisfying (26), as long as the realization of $\overline{\Psi}_\mathrm{d}$ in (16) is balanced. Based on the discussion above, we construct P in (21) by adding the basis of v to U in (23). More specifically, using the Gram-Schmidt orthogonalization, we find $P \in \mathbb{R}^{\nu \times n}$ such that

$$\operatorname{im}(P^\mathsf{T}) = \operatorname{im}(U^\mathsf{T}, v_1, \ldots, v_l), \quad PP^\mathsf{T} = I_\nu \tag{27}$$

where $v_1, \ldots, v_l$ denote the set of all v satisfying (15). This P appropriately ensures the stability of e. In addition, owing to the inclusion $\operatorname{im}(U^\mathsf{T}) \subseteq \operatorname{im}(P^\mathsf{T})$, we can expect that the approximation error in (20) is sufficiently small, as long as the value in (24) is small.

IV. Application to Power Networks

A. Power Network Model

In this section, we show the efficiency of the proposed hierarchical decentralized observer by an illustrative example of power networks. We deal with a power network model [11] composed of N subnetworks (subsystems), where the αth subsystem consists of $n^\mathrm{G}_{[\alpha]}$ generators and $n^\mathrm{L}_{[\alpha]}$ loads.

For $i \in \mathbb{N}^{\mathrm{G}}_{[\alpha]} := \{1,\ldots,n^{\mathrm{G}}_{[\alpha]}\}$, the dynamics of the ith generator is described by

$$\Sigma^{\mathrm{G}}_{[\alpha]i} : \left\{ \begin{array}{rcl} \dot{\phi}_{[\alpha]i} &=& A^{\mathrm{G}}_{[\alpha]i}\phi_{[\alpha]i} + \frac{1}{M^{\mathrm{G}}_{[\alpha]i}} b^{\mathrm{G}}\theta^{\mathrm{G}}_{[\alpha]i} + \frac{1}{k^{\mathrm{G}}_{[\alpha]i}} bu_{[\alpha]} \\ \delta^{\mathrm{G}}_{[\alpha]i} &=& c^{\mathrm{G}}\phi_{[\alpha]i} \end{array} \right. \tag{28}$$

where the states of $\phi_{[\alpha]i} \in \mathbb{R}^4$ denote phase angle difference, angular velocity difference, mechanical input difference, and valve position difference, and $\theta^{\mathrm{G}}_{[\alpha]i} \in \mathbb{R}$, $\delta^{\mathrm{G}}_{[\alpha]i} \in \mathbb{R}$, and $u_{[\alpha]} \in \mathbb{R}$ denote electric output difference, phase angle difference, and the command of angular velocity difference, respectively. Furthermore, the system matrices in (28) are given by

$$A^{\mathrm{G}}_{[\alpha]i} := \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & -D^{\mathrm{G}}_{[\alpha]i}/M^{\mathrm{G}}_{[\alpha]i} & -1/^{\mathrm{G}}_{[\alpha]i} & 0 \\ 0 & 0 & -1/T^{\mathrm{G}}_{[\alpha]i} & 1/T^{\mathrm{G}}_{[\alpha]i} \\ 0 & 1/k^{\mathrm{G}}_{[\alpha]i} & 0 & -R^{\mathrm{G}}_{[\alpha]i}/k^{\mathrm{G}}_{[\alpha]i} \end{array} \right]$$

and

$$b^{\mathrm{G}} := e^4_2, \quad c^{\mathrm{G}} := (e^4_1)^{\mathsf{T}}, \quad b = e^4_4$$

where $M^{\mathrm{G}}_{[\alpha]i}$, $D^{\mathrm{G}}_{[\alpha]i}$, $T^{\mathrm{G}}_{[\alpha]i}$, $k^{\mathrm{G}}_{[\alpha]i}$, and $R^{\mathrm{G}}_{[\alpha]i}$ denote positive constants that represent inertia constant, damping coefficient, turbine time constant, governor time constant, and velocity tuning rate, respectively, and $e^n_i \in \mathbb{R}^n$ denotes the ith column of I_n.

In a similar fashion, for $i \in \mathbb{N}^{\mathrm{L}}_{[\alpha]} := \{1,\ldots,n^{\mathrm{L}}_{[\alpha]}\}$, the dynamics of the ith load is described by

$$\Sigma^{\mathrm{L}}_{[\alpha]i} : \left\{ \begin{array}{rcl} \dot{\psi}_{[\alpha]i} &=& A^{\mathrm{L}}_{[\alpha]i}\psi_{[\alpha]i} + \frac{1}{M^{\mathrm{L}}_{[\alpha]i}} b^{\mathrm{L}}\theta^{\mathrm{L}}_{[\alpha]i} \\ \delta^{\mathrm{L}}_{[\alpha]i} &=& c^{\mathrm{L}}\psi_{[\alpha]i} \end{array} \right. \tag{29}$$

where each state of $\psi_{[\alpha]i} \in \mathbb{R}^2$ denotes phase angle difference and angular velocity difference, and $\theta^{\mathrm{L}}_{[\alpha]i} \in \mathbb{R}$ and $\delta^{\mathrm{L}}_{[\alpha]i} \in \mathbb{R}$ denote electric output difference and phase angle difference, respectively. Furthermore, the system matrices in (29) are given by

$$A^{\mathrm{L}}_{[\alpha]i} := \left[\begin{array}{cc} 0 & 1 \\ 0 & -D^{\mathrm{L}}_{[\alpha]i}/M^{\mathrm{L}}_{[\alpha]i} \end{array} \right], \quad b^{\mathrm{L}} := e^2_2, \quad c^{\mathrm{L}} := (e^2_1)^{\mathsf{T}}$$

where $M^{\mathrm{L}}_{[\alpha]i}$ and $D^{\mathrm{L}}_{[\alpha]i}$ denote positive constants that represent inertia constant and damping coefficient, respectively.

The interconnection structure among generators and loads are given by

$$\theta = -Y\delta, \quad \left\{ \begin{array}{l} \theta := [(\theta^{\mathrm{G}}_{[1]})^{\mathsf{T}}, (\theta^{\mathrm{L}}_{[1]})^{\mathsf{T}}, \ldots, (\theta^{\mathrm{G}}_{[N]})^{\mathsf{T}}, (\theta^{\mathrm{L}}_{[N]})^{\mathsf{T}}]^{\mathsf{T}} \\ \delta := [(\delta^{\mathrm{G}}_{[1]})^{\mathsf{T}}, (\delta^{\mathrm{L}}_{[1]})^{\mathsf{T}}, \ldots, (\delta^{\mathrm{G}}_{[N]})^{\mathsf{T}}, (\delta^{\mathrm{L}}_{[N]})^{\mathsf{T}}]^{\mathsf{T}} \end{array} \right. \tag{30}$$

where $Y \in \mathbb{R}^{N_Y \times N_Y}$ represents an admittance matrix satisfying

$$Y = Y^{\mathsf{T}}, \quad Y\mathbf{1}_{N_Y} = 0, \quad \left\{ \begin{array}{rcl} \mathbf{1}_n &:=& [1,\ldots,1]^{\mathsf{T}} \in \mathbb{R}^n \\ N_Y &:=& \sum_{\alpha=1}^N n^{\mathrm{G}}_{[\alpha]i} + n^{\mathrm{L}}_{[\alpha]i}, \end{array} \right.$$

and

$$\begin{array}{l} \theta^{\star}_{[\alpha]} := [\theta^{\star}_{[\alpha]1}, \ldots, \theta^{\star}_{[\alpha]n^{\star}_{[\alpha]}}]^{\mathsf{T}} \\ \delta^{\star}_{[\alpha]} := [\delta^{\star}_{[\alpha]1}, \ldots, \delta^{\star}_{[\alpha]n^{\star}_{[\alpha]}}]^{\mathsf{T}}, \end{array} \quad \star \in \{\mathrm{G}, \mathrm{L}\}.$$

We define the state as $x := [\phi^{\mathsf{T}}_{[1]}, \psi^{\mathsf{T}}_{[1]}, \ldots, \phi^{\mathsf{T}}_{[N]}, \psi^{\mathsf{T}}_{[N]}]^{\mathsf{T}}$ where

$$\begin{array}{l} \phi_{[\alpha]} := [(\phi_{[\alpha]1})^{\mathsf{T}}, \ldots, (\phi_{[\alpha]n^{\mathrm{G}}_{[\alpha]}})^{\mathsf{T}}]^{\mathsf{T}} \\ \psi_{[\alpha]} := [(\psi_{[\alpha]1})^{\mathsf{T}}, \ldots, (\psi_{[\alpha]n^{\mathrm{L}}_{[\alpha]}})^{\mathsf{T}}]^{\mathsf{T}}. \end{array}$$

Furthermore, we define the input u by $u := [u^{\mathsf{T}}_{[1]}, \ldots, u^{\mathsf{T}}_{[N]}]^{\mathsf{T}}$ as well as the outputs y and r by

$$y := \left[\begin{array}{c} \phi^1_{[1]} \\ \vdots \\ \phi^1_{[N]} \end{array} \right], \quad \phi^1_{[\alpha]} := \left[\begin{array}{c} \phi^1_{[\alpha]1} \\ \vdots \\ \phi^1_{[\alpha]n^{\mathrm{G}}_{[\alpha]}} \end{array} \right], \quad r := Fy \tag{31}$$

where $\phi^1_{[\alpha]i} \in \mathbb{R}$ denotes the first element of $\phi_{[\alpha]i} \in \mathbb{R}^4$ and F is a matrix having a dimension compatible with y. In this notation, for Σ in (3), the system matrices of the whole power network is given by (32), where $\otimes$ and $0_{n\times m} \in \mathbb{R}^{n\times m}$ denote the Kronecker product and the zero matrix, respectively.

B. Numerical Example

In what follows, we consider the state estimation problem for a power network model composed of 5 subsystems. We suppose that the dynamics of all generators and loads are respectively identical and their parameters are chosen as being compatible with physical models. Furthermore, the admittance matrix Y in (30) is given as the graph Laplacian of a complex network model, called the Holme-Kim model [12]. As the result, we obtain an 600-dimensional power network model, i.e., $n = 600$.

As shown in (28), an input $u_{[\alpha]}$, which denotes the command of angle velocity difference, is identically given to all generators belonging to the αth subsystem. Furthermore, as shown in (31), from each subsystem, we obtain the sensor signal $y_\alpha = \phi^1_{[\alpha]} \in \mathbb{R}^{n^{\mathrm{G}}_{[\alpha]}}$, which denotes the phase angle difference of generators. In particular, we obtain the additional sensor signal $r = \phi^1_{[1]} \in \mathbb{R}^{n^{\mathrm{G}}_{[1]}}$, used in the dynamical compensator Ψ in (4). By using these sensor signals, a hierarchical decentralized observer produces the estimated value $\hat{\phi}_{[\alpha]} \in \mathbb{R}^{4n^{\mathrm{G}}_{[\alpha]}}$ of generator states and $\hat{\psi}_{[\alpha]} \in \mathbb{R}^{2n^{\mathrm{L}}_{[\alpha]}}$ of load states.

In this setting, we demonstrate the efficiency of the proposed design method. Fig. 2 shows the trajectory of loads (the solid lines), where the input signal u and the initial state x_0 are given randomly. In this figure, for simplicity of depiction, we only show a part of load states in the fifth subsystem.

First, we use the 600-dimensional compensator in Theorem 1, i.e., $\nu = n$. We calculate some feedback gains $\{h_i\}_{i\in\mathbb{N}}$ and H such that $A_{i,i} - h_iC_i$ and $A - HS$ are stable, and desirable performance is achieved by the hierarchical decentralized observer. From Fig. 2, we can see that the estimated value (the bold lines) appropriately converges to the trajectory of loads (the solid lines). It should be noted that the system behavior for the input signal is exactly captured by the hierarchical decentralized observer, i.e., $\|\Delta\|_{\mathcal{H}_\infty} = 0$ for (17).

$$A = \operatorname*{diag}_{\alpha\in\mathbb{N}}\left(\begin{bmatrix} \operatorname*{diag}_{i\in\mathbb{N}^{\rm G}_{[\alpha]}}(A^{\rm G}_{[\alpha]i}) & \\ & \operatorname*{diag}_{i\in\mathbb{N}^{\rm L}_{[\alpha]}}(A^{\rm L}_{[\alpha]i}) \end{bmatrix}\right) - \operatorname*{diag}_{\alpha\in\mathbb{N}}\left(\begin{bmatrix} \operatorname*{diag}_{i\in\mathbb{N}^{\rm G}_{[\alpha]}}\left(\frac{1}{M^{\rm G}_{[\alpha]i}}b^{\rm G}\right) & \\ & \operatorname*{diag}_{i\in\mathbb{N}^{\rm L}_{[\alpha]}}\left(\frac{1}{M^{\rm L}_{[\alpha]i}}b^{\rm L}\right) \end{bmatrix}\right) Y \operatorname*{diag}_{\alpha\in\mathbb{N}}\left(\begin{bmatrix} \operatorname*{diag}_{i\in\mathbb{N}^{\rm G}_{[\alpha]}}(c^{\rm G}) & \\ & \operatorname*{diag}_{i\in\mathbb{N}^{\rm L}_{[\alpha]}}(c^{\rm L}) \end{bmatrix}\right)$$

$$B = \operatorname*{diag}_{\alpha\in\mathbb{N}}\left(\begin{bmatrix} \mathbf{1}_{n^{\rm G}_{[\alpha]}} \otimes b \\ 0_{2n^{\rm L}_{[\alpha]}\times 1} \end{bmatrix}\right), \qquad C = \operatorname*{diag}_{\alpha\in\mathbb{N}}\left(\begin{bmatrix} I_{4n^{\rm G}_{[\alpha]}} & 0_{4n^{\rm G}_{[\alpha]}\times 2n^{\rm L}_{[\alpha]}} \end{bmatrix}\right), \qquad S = FC \tag{32}$$

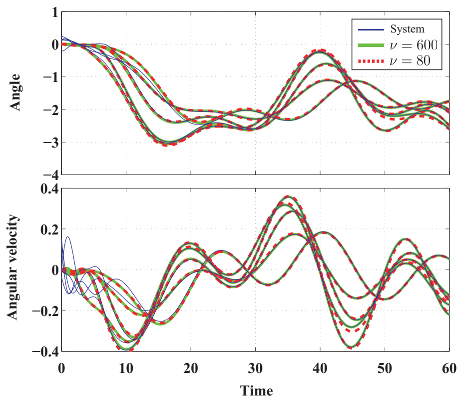

Fig. 2. Hierarchical Decentralized Estimation with 600- and 80-Dimensional Compensator.

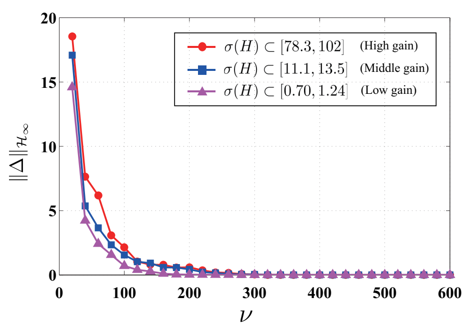

Fig. 3. Value of Approximation Error versus Dimension of Compensator.

Next, using the balanced truncation-based approximation, we design a lower-dimensional compensator. Note that, since this power network model has a zero-eigenvalue, the input weight V in (19) is not stable.

In Fig. 3, by the line with squares, we plot the value of $\|\Delta\|_{\mathcal{H}_\infty}$ in (17), i.e., a degree of performance degradation, against the dimension ν of the resultant dynamical compensator when using $\{h_i\}_{i\in\mathbb{N}}$ and H found above. From this figure, we can see that the performance of the hierarchical decentralized observer appropriately improves as increasing the dimension of the dynamical compensator. This result validates the efficiency of the proposed design method.

In Fig. 2, we also show the estimated value (the dotted lines) using the 80-dimensional compensator obtained by the approximation, i.e., $\nu = 80$. This result shows that the state estimation is achieved almost without causing errors, and that the proposed hierarchical decentralized observer is effective. In this case, the degree of performance degradation is $\|\Delta\|_{\mathcal{H}_\infty} = 2.41$.

Finally, for comparison, we show the values of $\|\Delta\|_{\mathcal{H}_\infty}$ when using H different in magnitude. We here denote the set of singular values of H by $\sigma(H)$. Fig. 3 also shows the values of $\|\Delta\|_{\mathcal{H}_\infty}$ by the lines with circles and triangles when using H such that $\sigma(H) \subset [78.3, 102]$ and $\sigma(H) \subset [0.70, 1.24]$, respectively. Note that the line with squares corresponds to H such that $\sigma(H) \subset [11.1, 13.5]$. We can see that the performance degradation tends to be larger as using *higher* feedback gains. Thus, this result reveals a trade-off relation between the performance degradation and the convergence rate of estimation errors.

V. Conclusion

In this paper, we have proposed a hierarchical decentralized observer for networked linear systems. The hierarchical decentralized observer is composed of a dynamical compensator and a set of decentralized observers. In particular, the dynamical compensator is systematically designed by using state-space expansion and model reduction. The efficiency of the proposed hierarchical decentralized observer has been fully demonstrated through an illustrative example of power networks. Our approach based on state-space expansion and model reduction can be expected as one effective approach to structured control system design.

References

[1] Y.-C. Ho and K.-C. Chu, "Team decision theory and information structures in optimal control problems–part I," *Automatic Control, IEEE Transactions on*, vol. 17, no. 1, pp. 15–22, 1972.

[2] M. Rotkowitz and S. Lall, "A characterization of convex problems in decentralized control," *Automatic Control, IEEE Transactions on*, vol. 51, no. 2, pp. 274–286, 2006.

[3] A. Monticelli, *State estimation in electric power systems: a generalized approach*. Springer, 1999, vol. 507.

[4] D. Šiljak and A. Zečević, "Control of large-scale systems: Beyond decentralized feedback," *Annual Reviews in Control*, vol. 29, no. 2, pp. 169–179, 2005.

[5] M. Ikeda, D. Šiljak, and D. White, "Decentralized control with overlapping information sets," *Journal of optimization theory and Applications*, vol. 34, no. 2, pp. 279–310, 1981.

[6] A. İftar, "Decentralized estimation and control with overlapping input, state, and output decomposition," *Automatica*, vol. 29, no. 2, pp. 511–516, 1993.

[7] S. S. Stankovic, M. J. Stanojevic, and D. D. Siljak, "Decentralized overlapping control of a platoon of vehicles," *Control Systems Technology, IEEE Transactions on*, vol. 8, no. 5, pp. 816–832, 2000.

[8] T. Ishizaki, Y. Sakai, K. Kashima, and J. Imura, "Hierarchical decentralized observer design for linearly coupled network systems," in *Decision and Control and European Control Conference (CDC-ECC) 2011 Proceedings of 50th IEEE Conference on*, 2011, pp. 7831–7836.

[9] H. Trinh and T. Fernando, *Functional observers for dynamical systems*. Springer Verlag, 2011, vol. 420.

[10] A. C. Antoulas, *Approximation of Large-Scale Dynamical Systems*. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics, 2005.

[11] M. D. Ilic and S. Liu, *Hierarchical power systems control: its value in a changing industry*. Springer Heidelberg, 1996.

[12] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, "Complex networks: Structure and dynamics," *Physics reports*, vol. 424, no. 4, pp. 175–308, 2006.