Monotonicity Characterization of Interval Quadratic Programming

Takayuki Ishizaki[1,2], Masakazu Koike[1,2], Nacim Ramdani[3], Jun-ichi Imura[1,2]

Abstract— **In this paper, we characterize a monotonicity of the solution of interval quadratic programming that involves an interval-valued parameter in the objective function and linear constraints. To give a characterization of monotonicity, we first analyze the derivative of a solution candidate for quadratic programming, on the basis of the Karush-Kuhn-Tucker condition. Then, deriving a condition for the objective function to assure the monotonicity, we propose a method to exactly find the upper and lower limits of optimal solutions by a finite number of operations. Finally, we provide an illustrative example of power supply scheduling to demonstrate the efficiency of our solution method.**

I. Introduction

The recent development of computing technology enables us to efficiently solve optimization problems in a numerical manner. In this trend, a wide variety of solution algorithms for linear and nonlinear programming have been investigated. In particular, for the past several decades, optimization problems involving interval-valued parameters have been gathering attention to deal with more realistic issues in which available data include uncertainty due to observation and prediction errors. In such optimization problems, interval analysis techniques are often utilized for, e.g., computing the enclosures of feasible solution space; see [1], [2] for an overview of interval analysis theory, and see [3], [4], [5] for the application of linear and nonlinear programming involving interval-valued parameters.

In this paper, we consider a quadratic programming problem that involves an interval-valued parameter, which we call *interval quadratic programming*. It is well known that quadratic programming is one of major optimization techniques in engineering and operations research, and it involves various kinds of important applications to engineering design [6], [7], economics [8] and so forth. Owing to the potential for wide applications, a number of efficient solution algorithms, such as interior-point methods [9], [10], have been developed for the standard quadratic programming. However, a solution to interval quadratic programming is not necessarily easy to find. This is because, to investigate the range of the optimal solution varying with an interval-valued parameter, we need to solve the quadratic programming for all possible (i.e., infinite many) parameters in general.

[1]Department of Mechanical and Environmental Informatics, Graduate School of Information Science and Engineering, Tokyo Institute of Technology; 2-12-1, Meguro, Tokyo, Japan:
{ishizaki, koike, imura}@cyb.mei.titech.ac.jp
[2]CREST, Japan Science and Technology Agency; 4-1-8, Honcho, Kawaguchi, Saitama, 332-0012, Japan.
[3]Universite d'Orleans, PRISME; 63 av. de Lattre de Tassigny, 18020 Bourges, France: Nacim.Ramdani@univ-orleans.fr

As one approach to overcome this difficulty, giving a characterization of the solution of interval quadratic programming, we propose a method to calculate the upper and lower limits of optimal solutions. It is turned out that the proposed method has the advantage to exactly find the upper and lower limits by a finite number of operations. Furthermore, to demonstrate its efficiency, we provide an illustrative example of power supply scheduling, in which the uncertainty of demand prediction is modeled as an interval-valued parameter.

Note that most of studies on interval analysis focuses mainly on finding the global extremum of a multivariable function, or on covering a feasible solution set complying with equality and inequality constraints. To this end, a constraint propagation technique [1], [2] is mostly applied. In fact, the straightforward application of interval arithmetics gives a solution to interval quadratic programming. However, it requires considerable computation cost to directly handle interval-valued parameters. Moreover, the resultant solution often becomes conservative due to an overestimation. To the best of authors' knowledge, there is no general method based on the interval analysis for solving interval quadratic programming in a computationally reasonable manner.

The reminder of this paper is structured as follows: In Section II, we first formulate an interval quadratic programming problem to find the lower and upper limits of optimal solutions. In Section III, we carry out a monotonicity analysis of the interval quadratic programming problem based on a representation of the solution candidate derived from the Karush-Kuhn-Tucker condition. By this analysis, we derive a characterization to guarantee the monotonicity of the optimal solution with respect to an interval-valued parameter. Then, in Section IV, we provide an illustrative example of power supply scheduling to demonstrate the efficiency of the proposed solution method. Finally, concluding remarks are provided in Section V.

Notation. The following notation is used in this paper:

$\mathbb{R}$	set of real numbers
$\mathbb{R}_{\geq 0}$	set of nonnegative real numbers
$\mathbb{R}_{>0}$	set of positive real numbers
I_n	n-dimensional unit matrix
e_i^n	ith column of I_n
$\mathbf{1}_n$	n-dimensional all-ones vector
$\lvert\mathcal{I}\rvert$	cardinality of a set $\mathcal{I}$
$\mathfrak{P}(\mathcal{I})$	power set of a set $\mathcal{I}$
$\mathrm{im}(M)$	image of a matrix M
$\ker(M)$	kernel of a matrix M
$\mathrm{rank}(M)$	rank of a matrix M

For a natural number n, let $\mathbb{N}[n] := \{1, \ldots, n\}$, and we denote the block-diagonal matrix having $M_1, \ldots, M_n$ on its block-diagonal by $\mathrm{diag}(M_i)_{i\in\mathbb{N}[n]}$. Furthermore, we denote a matrix composed of column vectors of I_n corresponding to the indices $\mathcal{I} \subseteq \mathbb{N}[n]$ by $e^n_{\mathcal{I}} \in \mathbb{R}^{n\times|\mathcal{I}|}$. In particular, if not confusing, we omit the superscript of n.

We define the sets of matrices

$$\begin{aligned}\mathbf{Z}_n &:= \{A \in \mathbb{R}^{n\times n} : a_{i,j} \le 0,\ \forall i, j \in \mathbb{N}[n],\ i \ne j\}\\ \mathbf{Z}^+_n &:= \{A \in \mathbf{Z}_n : a_{i,i} \ge 0,\ \forall i \in \mathbb{N}[n]\}\end{aligned}$$

where $a_{i,j}$ denotes the (i, j)-element of A. Furthermore, we define

$$\begin{aligned}\mathbf{M}_n &:= \{A \in \mathbf{Z}_n : A^{-1} \in \mathbb{R}^{n\times n}_{\ge 0}\}\\ \mathbf{M}^\dagger_n &:= \{A \in \mathbf{Z}_n : e^\mathsf{T}_{\mathcal{I}} A e_{\mathcal{I}} \in \mathbf{M}_{|\mathcal{I}|},\ \forall \mathcal{I} \in \mathfrak{P}(\mathbb{N}[n]) \backslash \{\emptyset, \mathbb{N}[n]\}\}\end{aligned}$$

and

$$\mathbf{R}_n := \{A \in \mathbb{R}^{n\times n}_{\ge 0} : (e^\mathsf{T}_{\mathcal{I}} A e_{\mathcal{I}})^{-1} \in \mathbf{Z}^+_n,\ \forall \mathcal{I} \in \mathfrak{P}(\mathbb{N}[n]) \backslash \emptyset\}.$$

II. Problem Formulation

In this paper, we deal with a quadratic programming problem given by

$$x^* = \arg\min_{x\in\mathbb{R}^n} J(x) \quad \text{s.t.} \quad \begin{cases} A_{\mathrm{in}} x \le b_{\mathrm{in}} \\ A_{\mathrm{eq}} x = b_{\mathrm{eq}} \end{cases} \tag{1}$$

where

$$A_{\mathrm{in}} \in \mathbb{R}^{m\times n}, \quad A_{\mathrm{eq}} \in \mathbb{R}^{r\times n}, \quad b_{\mathrm{in}} \in \mathbb{R}^m, \quad b_{\mathrm{eq}} \in \mathbb{R}^r$$

and

$$J(x) := \frac{1}{2} x^\mathsf{T} Q x - p^\mathsf{T} x \tag{2}$$

with a positive definite matrix $Q = Q^\mathsf{T} \in \mathbb{R}^{n\times n}$ and a vector $p \in \mathbb{R}^n$. Suppose that the constraints and objective function in (1) and (2) depend on an interval-valued parameter $d \in [\underline{d}, \overline{d}] \subseteq \mathbb{R}^\nu$. In this situation, the optimal solution x^* should be a function of the parameter d. For such quadratic programming, which called *interval quadratic programming* [11], [12], we formulate a problem to find the upper and lower limits of x^* as follows:

Problem 1: Given $[\underline{d}, \overline{d}] \subseteq \mathbb{R}^\nu$, define

$$\mathcal{X}^* := \{x^*(d) : d \in [\underline{d}, \overline{d}]\} \tag{3}$$

for an interval quadratic programming problem in (1). Then, find

$$\overline{x}^* = \{\overline{x}^*_i\} \in \mathbb{R}^n, \quad \underline{x}^* = \{\underline{x}^*_i\} \in \mathbb{R}^n \tag{4}$$

where $\overline{x}^*_i$ and $\underline{x}^*_i$ denote the maximum and minimum ith elements among all $x^* \in \mathcal{X}^*$.

In general, the solution of Problem 1 is not necessarily easy to obtain. This is because, to examine the upper and lower limits of x^*, we need to solve the interval quadratic programming for all possible parameters, i.e., infinite many parameters. In other words, solutions for a finite number of $d \in [\underline{d}, \overline{d}]$ do not give the exact bounds of x^*.

From a theoretical point of view, straightforward application of interval arithmetics can be used for computing the bounds of x^*. However, in practice, the resultant solution is often conservative since the interval arithmetics forces us an overestimation. Moreover, it generally requires considerable computation cost.

In view of this, as a key notion to efficiently solve Problem 1, we introduce the monotonicity of x^* as follows:

Definition 1: Let $[\underline{d}, \overline{d}] \subseteq \mathbb{R}^\nu$ be given. A function $f : \mathbb{R}^\nu \mapsto \mathbb{R}^n$ is said to be *monotone* with respect to d if, for any $i \in \mathbb{N}[\nu]$ and $j \in \mathbb{N}[\nu]$, there exists a constant $\sigma_{i,j} \in \{-1, 1\}$ such that

$$\sigma_{i,j} \frac{\partial f_j(d)}{\partial d_i} \ge 0, \quad \forall d \in [\underline{d}, \overline{d}] \subseteq \mathbb{R}^\nu \tag{5}$$

where f_i and d_i denote the ith elements of f and d, respectively. Furthermore, an interval quadratic programming problem in (1) is said to be *monotone* with respect to d if x^* is monotone with respect to d.

If an interval quadratic programming problem in (1) is monotone with respect to d, then the solution of Problem 1 is obtained by

$$\overline{x}^*_i = x^*_i(\overline{d}^{(i)}), \quad \underline{x}^*_i = x^*_i(\underline{d}^{(i)}) \tag{6}$$

where the jth elements of $\overline{d}^{(i)} \in \mathbb{R}^n$ and $\underline{d}^{(i)} \in \mathbb{R}^n$ are defined as

$$\begin{cases} \overline{d}^{(i)}_j := \sigma_{i,j} \max\left\{\sigma_{i,j}\underline{d}_j, \sigma_{i,j}\overline{d}_j\right\} \\ \underline{d}^{(i)}_j := \sigma_{i,j} \min\left\{\sigma_{i,j}\underline{d}_j, \sigma_{i,j}\overline{d}_j\right\}. \end{cases} \tag{7}$$

Note, however, that a mathematical characterization of the monotonicity is not trivial in general.

III. Monotonicity Characterization of Interval Quadratic Programming

A. Interval Quadratic Programming without Equality Constraint

First of all, we consider the case where the inequality constraint in (1) is solely imposed on the interval quadratic programming. A generalization to handle the equality constraint will be provided in Section III-B.

For the quadratic programming in (1), the Karush-Kuhn-Tucker condition assures that there exists a nonnegative $\lambda \in \mathbb{R}^m_{\ge 0}$ such that

$$\begin{cases} Qx^* - p + A^\mathsf{T}_{\mathrm{in}}\lambda = 0 \\ \lambda^\mathsf{T}(A_{\mathrm{in}}x^* - b_{\mathrm{in}}) = 0 \\ A_{\mathrm{in}}x^* - b_{\mathrm{in}} \le 0. \end{cases} \tag{8}$$

The following proposition gives a representation of x^* based on the Karush-Kuhn-Tucker condition:

Proposition 1: Consider an interval quadratic programming problem in (1). Let $\mathcal{I} \in \mathfrak{P}(\mathbb{N}[m])$ such that

$$\mathrm{rank}(A_{\mathcal{I}}) = |\mathcal{I}| \tag{9}$$

where $A_{\mathcal{I}} := e^\mathsf{T}_{\mathcal{I}} A_{\mathrm{in}} \in \mathbb{R}^{|\mathcal{I}|\times n}$, and define

$$x^{\mathrm{c}}(\mathcal{I}) := \begin{bmatrix} I_n & 0 \end{bmatrix} \begin{bmatrix} Q & A^\mathsf{T}_{\mathcal{I}} \\ A_{\mathcal{I}} & 0 \end{bmatrix}^{-1} \begin{bmatrix} p \\ b_{\mathcal{I}} \end{bmatrix} \tag{10}$$

where $b_{\mathcal{I}} := e^\mathsf{T}_{\mathcal{I}} b_{\mathrm{in}} \in \mathbb{R}^{|\mathcal{I}|}$. Then, it follows that

$$x^{\mathrm{c}}(\mathcal{I}^*) = x^*, \quad \mathcal{I}^* := \{i \in \mathbb{N}[m] : \lambda_i > 0\} \tag{11}$$

where λ_i denotes the ith element of $\lambda \in \mathbb{R}^m_{\geq 0}$ in (8).

Proof: Using the property of partitioned matrix inverse, we rewrite (10) by

$$x^{\rm c}(\mathcal{I}) = P_{\mathcal{I}} b_{\mathcal{I}} + G_{\mathcal{I}} p \tag{12}$$

where

$$\begin{array}{l} P_{\mathcal{I}} := Q^{-1} A_{\mathcal{I}}^{\mathsf{T}} (A_{\mathcal{I}} Q^{-1} A_{\mathcal{I}}^{\mathsf{T}})^{-1} \in \mathbb{R}^{n\times|\mathcal{I}|}, \\ G_{\mathcal{I}} := Q^{-1} - P_{\mathcal{I}} A_{\mathcal{I}} Q^{-1} \in \mathbb{R}^{n\times n}. \end{array} \tag{13}$$

Note that

$$e_{\mathcal{I}} e_{\mathcal{I}}^{\mathsf{T}} \lambda = \lambda, \quad A_{\mathcal{I}} x^* = b_{\mathcal{I}}$$

if $\mathcal{I} = \mathcal{I}^*$. Thus

$$x^* = Q^{-1} p - Q^{-1} A_{\mathcal{I}}^{\mathsf{T}} \xi \tag{14}$$

for $\xi := e_{\mathcal{I}}^{\mathsf{T}} \lambda$. Multiplying it by $A_{\mathcal{I}}$ from the left side, we have

$$b_{\mathcal{I}} = A_{\mathcal{I}} Q^{-1} p - A_{\mathcal{I}} Q^{-1} A_{\mathcal{I}}^{\mathsf{T}} \xi,$$

where $A_{\mathcal{I}} Q^{-1} A_{\mathcal{I}}^{\mathsf{T}}$ is nonsingular owing to (9). By solving this equation with respect to ξ and substituting it into (14), we obtain

$$x^* = P_{\mathcal{I}} b_{\mathcal{I}} + G_{\mathcal{I}} p,$$

which implies that $x^{\rm c}(\mathcal{I}^*) = x^*$ for (12). Hence, the claim follows. ∎

Proposition 1 gives a representation of the solution candidate $x^{\rm c}$ that is parametrized by the index set $\mathcal{I}$. It should be noted that the monotonicity of x^* is guaranteed if $x^{\rm c}$ is monotone for any $\mathcal{I} \in \mathfrak{P}(\mathbb{N}[m])$ satisfying (9). In what follows, for simplicity of arguments, we assume that the inequality constraint is given by

$$A_{\rm in} = \begin{bmatrix} I_n \\ -I_n \end{bmatrix}, \quad b_{\rm in} = \begin{bmatrix} b^+ \\ -b^- \end{bmatrix} \tag{15}$$

and the dependence on the parameter $d \in [\underline{d}, \overline{d}] \subseteq \mathbb{R}^\nu$ is represented as

$$\frac{\partial b^+(d)}{\partial d} = \frac{\partial b^-(d)}{\partial d} = \delta_1 e^n_{\mathbb{N}[\nu]}, \quad \frac{\partial p(d)}{\partial d} = \delta_2 e^n_{\mathbb{N}[\nu]} \tag{16}$$

for $\delta_1 \geq 0$ and $\delta_2 \geq 0$. In this setting, we derive a tractable representation of the derivative of $x^{\rm c}$ as follows:

Lemma 1: Consider an interval quadratic programming problem in (1) with (15), and suppose that (16) holds. Furthermore, suppose that no equality condition is imposed on (1). Let $\mathcal{I} \in \mathfrak{P}(\mathbb{N}[2n])$ such that (9), and define

$$\mathcal{K} := \mathcal{K}^+ \cup \mathcal{K}^- \in \mathfrak{P}(\mathbb{N}[n]) \tag{17}$$

where

$$\begin{cases} \mathcal{K}^+ := \{i \in \mathbb{N}[n] : i \in \mathcal{I}\} \\ \mathcal{K}^- := \{i \in \mathbb{N}[n] : (i+n) \in \mathcal{I}\}. \end{cases}$$

Then, $x^{\rm c}$ in (10) satisfies

$$\frac{\partial x^{\rm c}}{\partial d} = \left\{ \delta_1 \Pi(\mathcal{K}; Q^{-1}) + \delta_2 \Gamma(\mathcal{K}; Q^{-1}) \right\} e^n_{\mathbb{N}[\nu]} \tag{18}$$

where

$$\begin{array}{l} \Pi(\mathcal{K}; A) := A e_{\mathcal{K}} (e_{\mathcal{K}}^{\mathsf{T}} A e_{\mathcal{K}})^{-1} e_{\mathcal{K}}^{\mathsf{T}} \in \mathbb{R}^{n\times n} \\ \Gamma(\mathcal{K}; A) := A - \Pi(\mathcal{K}; A) A \in \mathbb{R}^{n\times n}. \end{array} \tag{19}$$

Proof: Owing to $\mathcal{I} \in \mathfrak{P}(\mathbb{N}[2n])$ satisfying (9), we verify that

$$\mathcal{K}^+ \cap \mathcal{K}^- = \emptyset,$$

which implies that

$$e_{\mathcal{I}} = \begin{bmatrix} e_{\mathcal{K}^+} & 0 \\ 0 & e_{\mathcal{K}^-} \end{bmatrix} \in \mathbb{R}^{2n\times(|\mathcal{K}^+|+|\mathcal{K}^-|)}$$

for (17). By defining

$$J_1 := \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}, \quad \tilde{J}_1 := \begin{bmatrix} I_{|\mathcal{K}^+|} & 0 \\ 0 & -I_{|\mathcal{K}^-|} \end{bmatrix},$$

we have $e_{\mathcal{I}}^{\mathsf{T}} J_1 = \tilde{J}_1 e_{\mathcal{I}}^{\mathsf{T}}$. Furthermore, we have

$$e_{\mathcal{I}}^{\mathsf{T}} J_2 = [e_{\mathcal{K}^+}, e_{\mathcal{K}^-}]^{\mathsf{T}} = e_{\mathcal{K}}^{\mathsf{T}}, \quad J_2 := \begin{bmatrix} I_n \\ I_n \end{bmatrix}.$$

Thus, it follows that

$$A_{\mathcal{I}} = e_{\mathcal{I}}^{\mathsf{T}} A_{\rm in} = e_{\mathcal{I}}^{\mathsf{T}} \tilde{J}_1 J_2 = \tilde{J}_1 e_{\mathcal{K}}^{\mathsf{T}}. \tag{20}$$

In addition, owing to (16), we have

$$\frac{\partial b_{\mathcal{I}}}{\partial d} = e_{\mathcal{I}}^{\mathsf{T}} \frac{\partial b_{\rm in}}{\partial d} = \delta_1 e_{\mathcal{I}}^{\mathsf{T}} \tilde{J}_1 J_2 e^n_{\mathbb{N}[\nu]} = \delta_1 \tilde{J}_1 e_{\mathcal{K}}^{\mathsf{T}} e^n_{\mathbb{N}[\nu]}.$$

Thus, by (12), we obtain

$$\frac{\partial x^{\rm c}}{\partial d} = \left(\delta_1 P_{\mathcal{I}} \tilde{J}_1 e_{\mathcal{K}}^{\mathsf{T}} + \delta_2 G_{\mathcal{I}} \right) e^n_{\mathbb{N}[\nu]}.$$

Finally, substituting (20) into $P_{\mathcal{I}}$ and $G_{\mathcal{I}}$ yields

$$P_{\mathcal{I}} \tilde{J}_1 e_{\mathcal{K}}^{\mathsf{T}} = \Pi(\mathcal{K}; Q^{-1}), \quad G_{\mathcal{I}} = \Gamma(\mathcal{K}; Q^{-1}).$$

Hence, the claim follows. ∎

Lemma 1 shows that the derivative of $x^{\rm c}$ defined as in (10) can be represented as a function of Π and Γ in (19). The sign pattern of the matrix functions Π and Γ can be analyzed as follows:

Lemma 2: Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. Then, for any $\mathcal{K} \in \mathfrak{P}(\mathbb{N}[n])$, Π and Γ in (19) satisfy

$$\Pi(\mathcal{K}; A) \in \mathbf{Z}^+_n, \quad A \in \mathbf{M}_n$$

and

$$\Gamma(\mathcal{K}; A) \in \begin{cases} \mathbf{Z}^+_n, & A^{-1} \in \mathbf{R}_n \\ \mathbb{R}^{n\times n}_{\geq 0}, & A^{-1} \in \mathbf{M}_n. \end{cases}$$

Proof: First, from the structure of Π, it follows that

$$e_{\mathcal{K}}^{\mathsf{T}} \Pi e_{\mathcal{K}} = I_{|\mathcal{K}|}, \quad e_{\overline{\mathcal{K}}}^{\mathsf{T}} \Pi e_{\overline{\mathcal{K}}} = 0, \quad e_{\mathcal{K}}^{\mathsf{T}} \Pi e_{\overline{\mathcal{K}}} = 0 \tag{21}$$

where $\overline{\mathcal{K}} := \mathbb{N}[n] \backslash \mathcal{K}$. Note that, if $A \in \mathbf{M}_n$, it follows that

$$-e_{\overline{\mathcal{K}}}^{\mathsf{T}} A e_{\mathcal{K}} \in \mathbb{R}^{|\overline{\mathcal{K}}|\times|\mathcal{K}|}_{\geq 0}, \quad (e_{\mathcal{K}}^{\mathsf{T}} A e_{\mathcal{K}})^{-1} \in \mathbb{R}^{|\mathcal{K}|\times|\mathcal{K}|}_{\geq 0}.$$

Thus

$$e_{\overline{\mathcal{K}}}^{\mathsf{T}} \Pi(\mathcal{K}; A) e_{\mathcal{K}} = e_{\overline{\mathcal{K}}}^{\mathsf{T}} A e_{\mathcal{K}} (e_{\mathcal{K}}^{\mathsf{T}} A e_{\mathcal{K}})^{-1}$$

is constructed by the product of nonpositive and nonnegative matrices. This implies that all elements of the submatrix $e_{\overline{\mathcal{K}}}^{\mathsf{T}} \Pi e_{\mathcal{K}}$ are nonpositive. Hence, in conjunction with (21), $\Pi \in \mathbf{Z}^+_n$ is proven for any $\mathcal{K} \in \mathfrak{P}(\mathbb{N}[n])$.

On the other hand, applying Lemma 6 in Appendix to Π, we have

$$\Gamma(\mathcal{K}; A) = e_{\overline{\mathcal{K}}} (e_{\overline{\mathcal{K}}}^{\mathsf{T}} A^{-1} e_{\overline{\mathcal{K}}})^{-1} e_{\overline{\mathcal{K}}}^{\mathsf{T}}.$$

Thus, the submatrices of Γ satisfy

$$e_{\mathcal{K}}^{\mathsf{T}}\Gamma e_{\mathcal{K}} = 0, \quad e_{\mathcal{K}}^{\mathsf{T}}\Gamma e_{\overline{\mathcal{K}}} = 0, \quad e_{\overline{\mathcal{K}}}^{\mathsf{T}}\Gamma e_{\mathcal{K}} = 0.$$

By noting that

$$e_{\overline{\mathcal{K}}}^{\mathsf{T}}\Gamma(\mathcal{K};A)e_{\overline{\mathcal{K}}} \in \begin{cases} \mathbf{Z}_{|\overline{\mathcal{K}}|}^{+}, & A^{-1}\in\mathbf{R}_n \\ \mathbb{R}_{\geq 0}^{|\overline{\mathcal{K}}|\times|\overline{\mathcal{K}}|}, & A^{-1}\in\mathbf{M}_n, \end{cases}$$

we verify the claim. ■

Lemma 2 shows that the sign pattern of Π and Γ is invariant with respect to any $\mathcal{K}\in\mathfrak{P}(\mathbb{N}[n])$ if the argument A of Π and Γ belongs to a particular class of matrices. Based on this fact, we can prove the following theorem:

Theorem 1: Consider an interval quadratic programming problem in (1) with (15), and suppose that (16) holds. Furthermore, suppose that no equality condition is imposed. If

$$\begin{cases} (Q\in\mathbf{R}_n)\vee(Q\in\mathbf{M}_n), & \delta_1=0 \\ Q^{-1}\in\mathbf{M}_n, & \delta_2=0 \\ (Q\in\mathbf{R}_n)\wedge(Q^{-1}\in\mathbf{M}_n), & \delta_1,\delta_2\neq 0, \end{cases} \tag{22}$$

then the quadratic programming problem is monotone with respect to d.

Proof: If (22), it follows from Lemma 2 that

$$\begin{cases} (\Gamma\in\mathbf{Z}_n^{+})\vee(\Gamma\in\mathbb{R}_{\geq 0}^{n\times n}), & \delta_1=0 \\ \Pi\in\mathbf{Z}_n^{+}, & \delta_2=0 \\ (\Pi\in\mathbf{Z}_n^{+})\wedge(\Gamma\in\mathbf{Z}_n^{+}), & \delta_1,\delta_2\neq 0. \end{cases}$$

Hence, the claim follows. ■

Theorem 1 shows that the interval quadratic programming in (1) possesses the monotonicity if the matrix Q, which corresponds to the quadratic part of the objective function in (2), belongs to the class of matrices specified by (22). As one of meaningful matrices satisfying (22), we introduce the following class of matrices:

Lemma 3: Given $\alpha\in(-\infty,1]$ and $w\in\mathbb{R}_{>0}^{n}$, define

$$H_n(\alpha,w) := I_n - \alpha\frac{ww^{\mathsf{T}}}{w^{\mathsf{T}}w}\in\mathbb{R}^{n\times n}. \tag{23}$$

Then, it follows that

$$H_n(\alpha,w)\in\begin{cases} \mathbf{R}_n, & \alpha\in(-\infty,0] \\ \mathbf{M}_n, & \alpha\in[0,1) \\ \mathbf{M}_n^{\dagger}, & \alpha=1 \end{cases} \tag{24}$$

and

$$H_n^{-1}(\alpha,w)\in\begin{cases} \mathbf{M}_n, & \alpha\in(-\infty,0] \\ \mathbf{R}_n, & \alpha\in[0,1). \end{cases} \tag{25}$$

Proof: From direct calculation, we can verify that

$$H_n^{-1}(\alpha,w) = H_n\left(\frac{\alpha}{\alpha-1},w\right). \tag{26}$$

First, if $\alpha\in[0,1)$, $H_n\in\mathbf{M}_n$ follows from $H_n\in\mathbf{Z}_n$ and $H_n^{-1}\in\mathbb{R}_{\geq 0}^{n\times n}$. Similarly, if $\alpha\in(-\infty,0]$, $H_n^{-1}\in\mathbf{M}_n$ follows from (26).

Next, if $\alpha=1$, the principal submatrix of the singular $H_n\in\mathbf{Z}_n$ can be represented by

$$e_{\mathcal{K}}^{\mathsf{T}}H_n(1,w)e_{\mathcal{K}}, \quad \mathcal{K}\in\mathfrak{P}(\mathbb{N}[n])\backslash\{\emptyset,\mathbb{N}[n]\}.$$

Then, we have

$$e_{\mathcal{K}}^{\mathsf{T}}H_n(1,w)e_{\mathcal{K}} = H_{|\mathcal{K}|}\left(\frac{w_{\mathcal{K}}^{\mathsf{T}}w_{\mathcal{K}}}{w^{\mathsf{T}}w},w_{\mathcal{K}}\right), \quad w_{\mathcal{K}} := e_{\mathcal{K}}^{\mathsf{T}}w.$$

Note that

$$0<\frac{w_{\mathcal{K}}^{\mathsf{T}}w_{\mathcal{K}}}{w^{\mathsf{T}}w}<1, \quad \forall\mathcal{K}\in\mathfrak{P}(\mathbb{N}[n])\backslash\{\emptyset,\mathbb{N}[n]\}$$

since all entries of w are nonzero. Thus, for any principal submatrix of H_n, $e_{\mathcal{K}}^{\mathsf{T}}H_n e_{\mathcal{K}}\in\mathbf{M}_{|\mathcal{K}|}$ holds. Hence, $H_n\in\mathbf{M}_n^{\dagger}$ follows.

Finally, if $\alpha\in(-\infty,0]$, the inverse of the principal submatrices of $H_n\in\mathbb{R}_{\geq 0}^{n\times n}$ satisfies

$$(e_{\mathcal{K}}^{\mathsf{T}}H_n(\alpha,w)e_{\mathcal{K}})^{-1} = H_{|\mathcal{K}|}^{-1}(\tilde{\alpha},w_{\mathcal{K}})\in\mathbf{M}_{|\mathcal{K}|}\subset\mathbf{Z}_{|\mathcal{K}|}$$

where

$$\tilde{\alpha} := \alpha\frac{w_{\mathcal{K}}^{\mathsf{T}}w_{\mathcal{K}}}{w^{\mathsf{T}}w}\in(-\infty,0].$$

This implies that $H_n\in\mathbf{R}_n$ holds for any $\mathcal{K}$. Similarly, if $\alpha\in[0,1)$, $H_n^{-1}\in\mathbf{R}_n$ is proven by (26). ■

Based on Lemma 3, the following result is straightforwardly obtained:

Corollary 1: Consider an interval quadratic programming problem in (1) with (15), and suppose that (16) holds. Furthermore, suppose that no equality condition is imposed. For a natural number l, let

$$Q = \operatorname{diag}\left(H_{n_i}(\alpha_i,w_i)\right)_{i\in\mathbb{N}[l]}, \quad \begin{cases} \alpha_i\in(-\infty,1) \\ w_i\in\mathbb{R}_{>0}^{n_i} \end{cases} \tag{27}$$

where $\sum_{i=1}^{l} n_i = n$ and H_{n_i} is defined as in (23). If

$$\begin{cases} \alpha_i\in(-\infty,1), & \forall i\in\mathbb{N}[l], & \delta_1=0 \\ \alpha_i\in(-\infty,0], & \forall i\in\mathbb{N}[l], & \delta_1\neq 0, \end{cases} \tag{28}$$

then the quadratic programming problem is monotone with respect to d.

Proof: If (28) holds, then (22) follows from Lemma 3. Hence, the claim follows. ■

From Corollary 1 with Lemma 3, we can see that, if $\alpha_i\in(-\infty,0]$ for all $i\in\mathbb{N}[l]$, both Π and Γ belong to the class of $\mathbf{Z}_n^{+}$. This implies that x_i^{*} is a monotonically increasing function of d_i and a monotonically decreasing function of d_j for $j\neq i$.

B. Interval Quadratic Programming with Equality Constraint

In this subsection, based on an orthogonal projection of the solution space, we convert a quadratic programing problem with an equality constraint to that without the equality constraint. We show that the monotonicity of an interval quadratic programing problem with equality constraints can be analyzed by the procedure same as in Section III-A.

Let $\eta\in\mathbb{R}^n$ be a vector such that $A_{\mathrm{eq}}\eta = b_{\mathrm{eq}}$ for (1). Then, by replacing $x-\eta$ by x as a new variable, we can rewrite the equality constraint as

$$A_{\mathrm{eq}}x = 0. \tag{29}$$

It should be noted that this change of variables does not affect the monotonicity of solutions as long as η is constant. Therefore, we can assume (29) without loss of generality. Based on this fact, we obtain the following lemma:

Lemma 4: Consider an interval quadratic programming problem in (1), and suppose that $b_{\rm eq} = 0$. Define $V \in \mathbb{R}^{n\times \hat{n}}$ such that

$$\mathrm{im}(V) = \ker(A_{\rm eq}), \quad V^{\mathsf T} V = I_{\hat{n}} \tag{30}$$

where $\hat{n} := n - r$. Then, x^* is given by

$$x^* = V\hat{x}^*, \quad \hat{x}^* := \arg\min_{\hat{x}\in\mathbb{R}^{\hat{n}}} \hat{J}(\hat{x}) \quad \text{s.t.} \quad A_{\rm in}V\hat{x} \leq b_{\rm in} \tag{31}$$

where

$$\hat{J}(\hat{x}) := \frac{1}{2}\hat{x}^{\mathsf T}\hat{Q}\hat{x} - \hat{p}^{\mathsf T}\hat{x}, \quad \left\{ \begin{array}{l} \hat{Q} := V^{\mathsf T} Q V \\ \hat{p} := V^{\mathsf T} p. \end{array}\right.$$

Proof: We notice that $VV^{\mathsf T}$ is the orthogonal projection matrix onto $\ker(A_{\rm eq})$ by the definition of (30). Thus

$$\mathcal{A} := VV^{\mathsf T} = I_n - A_{\rm eq}^{\mathsf T}(A_{\rm eq}A_{\rm eq}^{\mathsf T})^{-1}A_{\rm eq} \tag{32}$$

holds. Furthermore, $x \in \ker(A_{\rm eq})$ for $\hat{x} = V^{\mathsf T}x$, or equivalently, (29) holds if and only if

$$x = \mathcal{A}x = V\hat{x}.$$

Hence, the claim follows from rewriting (1) by using $\hat{x}$ as the variable of the quadratic programing in (31). ■

As shown in Lemma 4, we can equivalently rewrite a quadratic programming problem with an equality constraint by that without the equality constraint. Based on this fact, similarly to (18), we obtain the following representation of the derivative of $x^{\rm c}$ in (10):

Lemma 5: Consider an interval quadratic programming problem in (1) with (15), and suppose that (16) holds. Let $\mathcal{I} \in \mathfrak{P}(\mathbb{N}[2n])$ such that (9), and define $\mathcal{K}$ as in (17). Then, $x^{\rm c}$ in (10) satisfies

$$\frac{\partial x^{\rm c}}{\partial d} = \{\delta_1 \Pi(\mathcal{K}; Q_V) + \delta_2 \Gamma(\mathcal{K}; Q_V)\}\, e^n_{\mathbb{N}[\nu]} \tag{33}$$

where Π and Γ are defined as in (19) and

$$Q_V := V(V^{\mathsf T} Q V)^{-1} V^{\mathsf T} \in \mathbb{R}^{n\times n} \tag{34}$$

with V satisfying (30).

Proof: To show the claim by a manner similar to the proof of Lemma 1, we replace the symbols in Lemma 1 as

$$x^{\rm c} \to \hat{x}^{\rm c}, \quad A_{\rm in} \to A_{\rm in}V, \quad Q \to \hat{Q}, \quad p \to \hat{p}.$$

Then, from (12), we have

$$\frac{\partial \hat{x}^{\rm c}}{\partial d} = \left(\delta_1 \hat{P}_{\mathcal{I}} \tilde{J}_1 e_{\mathcal{K}}^{\mathsf T} + \delta_2 \hat{G}_{\mathcal{I}} V^{\mathsf T}\right) e^n_{\mathbb{N}[\nu]}$$

where $\hat{P}_{\mathcal{I}}$ and $\hat{G}_{\mathcal{I}}$ are defined by replacing the symbols in (13) as

$$Q \to \hat{Q} = V^{\mathsf T} Q V, \quad A_{\mathcal{I}} \to \hat{A}_{\mathcal{I}} = \tilde{J}_1 e_{\mathcal{K}}^{\mathsf T} V.$$

Noting that $V\hat{x}^{\rm c} = x^{\rm c}$, we obtain

$$V\hat{P}_{\mathcal{I}}\tilde{J}_1 e_{\mathcal{K}}^{\mathsf T} = \Pi(\mathcal{K}; Q_V), \quad V\hat{G}_{\mathcal{I}} V^{\mathsf T} = \Gamma(\mathcal{K}; Q_V).$$

Hence, the claim follows. ■

As shown in Lemma 5, the derivative of $x^{\rm c}$ can be represented by the functions of Π and Γ, similarly to the case where the inequality constraint is not imposed on the quadratic programming. It should be noted that Q^{-1} in (18) is replaced with Q_V in (33), which is defined through the orthogonal projection. Based on this fact, we can prove the following theorem:

Theorem 2: Consider an interval quadratic programming problem in (1) with (15), and suppose that (16) holds. Let $\mathcal{I} \in \mathfrak{P}(\mathbb{N}[2n])$ such that (9), and define $\mathcal{K}$ as in (17). If $Q = I_n$ and

$$A_{\rm eq} = \mathrm{diag}(w_i^{\mathsf T})_{i\in\mathbb{N}[r]} \in \mathbb{R}^{r\times n}_{\geq 0}, \quad w_i \in \mathbb{R}^{n_i}_{>0} \tag{35}$$

where $\sum_{i=1}^r n_i = n$, then the quadratic programming problem is monotone with respect to d.

Proof: Note that if $Q = I_n$, then $Q_V = \mathcal{A}$ for $\mathcal{A}$ defined as in (32). Furthermore, by the definition of $\mathcal{A}$ and Lemma 3, we verify that

$$\mathcal{A} = \mathrm{diag}\left(H_{n_i}(1, w_i)\right)_{i\in\mathbb{N}[r]} \in \mathbf{M}_n^{\dagger} \tag{36}$$

for $A_{\rm eq}$ in (35). Thus, in a manner similar to the proof of Lemma 2, we can prove that, for any $\mathcal{K}$ in (17) constructed by $\mathcal{I} \in \mathfrak{P}(\mathbb{N}[2n])$ satisfying (9), $\Pi(\mathcal{K}; \mathcal{A}) \in \mathbf{Z}_n^+$ holds.

On the other hand, substituting (32) into Γ yields

$$\Gamma(\mathcal{K}; \mathcal{A}) = e_{\overline{\mathcal{K}}}\mathrm{diag}\left(H_{n_i}(1, e_{\overline{\mathcal{K}}_i}^{\mathsf T} w_i)\right)_{i\in\mathbb{N}[r]} e_{\overline{\mathcal{K}}}^{\mathsf T} \in \mathbf{Z}_n^+$$

where

$$\overline{\mathcal{K}} := \mathbb{N}[n]\backslash\mathcal{K}, \quad e_{\overline{\mathcal{K}}} = [e_{\overline{\mathcal{K}}_1}, \ldots, e_{\overline{\mathcal{K}}_r}].$$

Hence, the claim follows. ■

By comparing Theorem 2 with Corollary 1, we can see that, from the viewpoint of monotonicity, the interval quadratic programming with $Q = I_n$ subject to the equality constraint in (36) is equivalent to that with Q given as in (27) for $\alpha_i \in (-\infty, 0]$ subject to no equality constraint. It should be noted that, if we get rid of the assumption of $Q = I_n$, Q_V in (34) does not coincide with $\mathcal{A}$ in (36), and thus it is difficult to systematically characterize the monotonicity of the interval quadratic programming.

IV. Application to Power Supply Scheduling

A. Monotonicity Analysis

In this subsection, we formulate a power supply scheduling problem as an interval quadratic programming problem. In this problem, taking into account the uncertainty of demand prediction, we aim to minimize an objective function reflecting the fuel cost of generators and the degradation cost of storage batteries.

Dividing a day into n moments, we denote the temporal sequence of uncertain predicted demand by $d \in \mathbb{R}^n$. We suppose that d varies within a fixed interval $[\underline{d}, \overline{d}] \subseteq \mathbb{R}^n$, which can be regarded as a confidence interval of demand prediction [13], [14]. For this demand prediction, we consider keeping a supply-demand balance by the power of generators and the charge and discharge power of storage batteries. We denote the temporal sequence of the total generator power by $x \in \mathbb{R}^n$, and the total charge and discharge power of storage batteries by $\Delta y \in \mathbb{R}^n$. In this notation, the supply-demand balance can be represented by

$$\Delta y = x - d. \tag{37}$$

We impose the inequality constraint described as

$$\Delta y_{\min} \mathbf{1}_n \le \Delta y \le \Delta y_{\max} \mathbf{1}_n \tag{38}$$

where $\Delta y_{\min} \in \mathbb{R}$ and $\Delta y_{\max} \in \mathbb{R}$ are the constants representing the bounds of Δy. Furthermore, we denote the total energy of the batteries by

$$y := [y_1, \dots, y_n]^\mathsf{T} \in \mathbb{R}^n \tag{39}$$

where the ith element is defined by

$$y_i := y_0 + \sum_{j=1}^{i} \Delta y_j$$

with the initial value of $y_0 \in \mathbb{R}$. The equality constraint is imposed on (39) as

$$y_n = y_0 + y_\mathrm{D} \tag{40}$$

where $y_n \in \mathbb{R}$ denotes the total energy at the termination time, and $y_\mathrm{D} \in \mathbb{R}$ denotes a desired energy to be charged in the day of interest. Moreover, taking into account the fuel cost of generators and the degradation cost of batteries, we define the objective function by

$$f(x, \Delta y) := \alpha_1 x^\mathsf{T} x + \alpha_2 \mathbf{1}_n^\mathsf{T} x + \alpha_3 \Delta y^\mathsf{T} \Delta y + \alpha_4 \mathbf{1}_n^\mathsf{T} \Delta y \tag{41}$$

where $\alpha_1, \dots \alpha_4 \in \mathbb{R}$ are nonnegative coefficients.

By substituting (37), the optimal power generation plan that minimizes the objective function in (41) is given by

$$x^*(d) = \arg\min_{x \in \mathbb{R}^n} f(x, x - d) \tag{42}$$

where the inequality constraint in (38) can be rewritten as

$$C_\mathrm{in}(x; d) : \begin{bmatrix} I_n \\ -I_n \end{bmatrix} x \le \begin{bmatrix} \Delta y_{\min} \mathbf{1}_n + d \\ -(\Delta y_{\max} \mathbf{1}_n + d) \end{bmatrix} \tag{43}$$

and the equality constraint in (40) as

$$C_\mathrm{eq}(x; d) : \mathbf{1}_n^\mathsf{T} x = \mathbf{1}_n^\mathsf{T} d + y_\mathrm{D}. \tag{44}$$

In addition, as being compatible with (37), the optimal battery charge cycles is given by

$$\Delta y^*(d) := x^*(d) - d. \tag{45}$$

For this interval quadratic programming, we can prove the following fact:

Corollary 2: Let $d \in [\underline{d}, \overline{d}]$ be given, and consider the interval quadratic programming in (42) subject to

$$C_\mathrm{in}(x; d), \quad C_\mathrm{eq}(x; d_0) \tag{46}$$

where C_in and C_eq are defined as in (43) and (44), respectively, and

$$d_0 := \frac{\underline{d} + \overline{d}}{2} \in \mathbb{R}^n. \tag{47}$$

Define Δy^* as in (45). Then, both x^* and Δy^* are monotone with respect to d.

Proof: Note that the objective function in (42) without the constant term can be rewritten by J in (2) with

$$Q = 2(\alpha_1 + \alpha_3) I_n, \quad p = 2\alpha_3 d - \alpha_2 \mathbf{1}_n.$$

Thus, by using Theorem 2, the monotonicity of x^* is proven. Next, we prove the monotonicity of Δy^*. To this end, it suffices to show that

$$\Delta y^\mathrm{c}(\mathcal{I}; d) := x^\mathrm{c}(\mathcal{I}; d) - d$$

is monotone for any $\mathcal{I} \in \mathfrak{P}(\mathbb{N}[2n])$ such that (9), where x^c is defined as in (10). By direct calculation, we obtain

$$\frac{\partial \Delta y^\mathrm{c}}{\partial d} = -(1 - \delta) e_{\overline{\mathcal{K}}} e_{\overline{\mathcal{K}}}^\mathsf{T} - \frac{1}{|\overline{\mathcal{K}}|} e_{\overline{\mathcal{K}}} \mathbf{1}_{|\overline{\mathcal{K}}|} \mathbf{1}_n^\mathsf{T} (e_\mathcal{K} e_\mathcal{K}^\mathsf{T} + \delta e_{\overline{\mathcal{K}}} e_{\overline{\mathcal{K}}}^\mathsf{T})$$

where $\mathcal{K}$ is defined as in (17), $\overline{\mathcal{K}} := \mathbb{N}[n] \backslash \mathcal{K}$ and

$$\delta := \frac{\alpha_3 \alpha_1}{\alpha_2 + \alpha_3 \alpha_1}.$$

Since $\delta \in [0, 1]$, all elements of $\partial \Delta y^\mathrm{c} / \partial d$ are nonpositive for any $\mathcal{K} \in \mathfrak{P}(\mathbb{N}[n])$. Hence, the claim follows. ■

Corollary 2 shows that the interval quadratic programming problem in (42) is monotone as long as the equality constraint does not depend on the parameter d. Moreover, in this power supply scheduling problem, not only the optimal power generation plan x^* but also the optimal battery charge cycles Δy^* possess the monotonicity with respect to d.

B. Numerical Example

In this subsection, through a numerical example, we show the efficiency of our method to find the lower and upper limits of optimal solutions. As a confidence interval of demand prediction, we use the temporal sequence of intervals shown in Fig. 1, which is based on the data provided by Tokyo Electric Power Company managing 19 million demanders. In this figure, the solid line with circles represents d_0 in (47), and the thick solid lines represent $\underline{d}$ and $\overline{d}$.

We fix the bounds of Δy in (38) as $\Delta y_{\max} = 6 \times 10^6$ kW and $\Delta y_{\min} = -6 \times 10^6$ kW, the initial and termination values of total energy as $y_0 = y_\mathrm{D} = 0$, and the coefficients of the fuel cost function in (41) as $\alpha_1 = 0.38 \times 10^{-12}$ JPY/W^{2}h and $\alpha_2 = 5000 \times 10^{-6}$ JPY/Wh. In this setting, varying the coefficients of the degradation cost of storage batteries as shown in Table I, we solve the interval quadratic programming problem in (42). The result is shown in Fig. 2, whose first to third figure corresponds to **Case 1** to **Case 3**, respectively. In these figures, the lines with circles denote the optimal solutions for d_0 in (47), and the thick solid lines denote the values of the upper and lower limits. From these figures, we see that the difference between the upper and lower limits of power generation plan, which corresponds to the regulating capacity of generators, increases as we use more expensive, i.e., higher degradation cost, storage batteries.

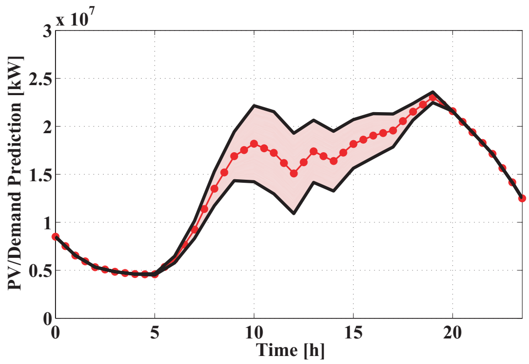

Fig. 1. Model of Uncertain Demand Prediction.

V. Conclusion

In this paper, we have characterized a monotonicity of the solution of interval quadratic programming that involves an interval-valued parameter in the objective function and linear constraints. Based on the monotonicity characterization, we have proposed a method to find the lower and upper limits of the solutions for the interval quadratic programming by a finite number of operations. Moreover, we have provided an illustrative example of power supply scheduling and we have shown that the optimal solution of power generation as well as battery charge cycles possess the monotonicity. An enhancement to handle more general form of inequality and equality constraints is currently under investigation.

Appendix

Lemma 6: Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix, and $P \in \mathbb{R}^{n\times k}$ and $\overline{P} \in \mathbb{R}^{n\times(n-k)}$ such that

$$PP^{\mathsf{T}} + \overline{P}\,\overline{P}^{\mathsf{T}} = I_n.$$

Suppose that $P^{\mathsf{T}}AP$ and $\overline{P}^{\mathsf{T}}A^{-1}\overline{P}$ are nonsingular. Then

$$AP(P^{\mathsf{T}}AP)^{-1}P^{\mathsf{T}} = I_n - \overline{P}(\overline{P}^{\mathsf{T}}A^{-1}\overline{P})^{-1}\overline{P}^{\mathsf{T}}A^{-1}$$

holds.

Proof: Using $\Xi := \overline{P}(\overline{P}^{\mathsf{T}}A^{-1}\overline{P})^{-1}\overline{P}^{\mathsf{T}}$, we define

$$Y := I_n - \Xi A^{-1} = I_n - \overline{P}(\overline{P}^{\mathsf{T}}A^{-1}\overline{P})^{-1}\overline{P}^{\mathsf{T}}A^{-1}.$$

Furthermore, by defining $X := I_n - A^{-1}\Xi$, it follows from $(A^{-1}\Xi)^2 = A^{-1}\Xi$ that

$$XA^{-1}\Xi = 0.$$

TABLE I

Coefficients of Degradation Cost Function of Storage Batteries.

	Case 1	Case 2	Case 3
α_3 JPY/W^{2}h	1.80×10^{-14}	1.80×10^{-13}	1.80×10^{-12}
α_4 JPY/Wh	-2.17×10^{-7}	-2.17×10^{-6}	-2.17×10^{-5}

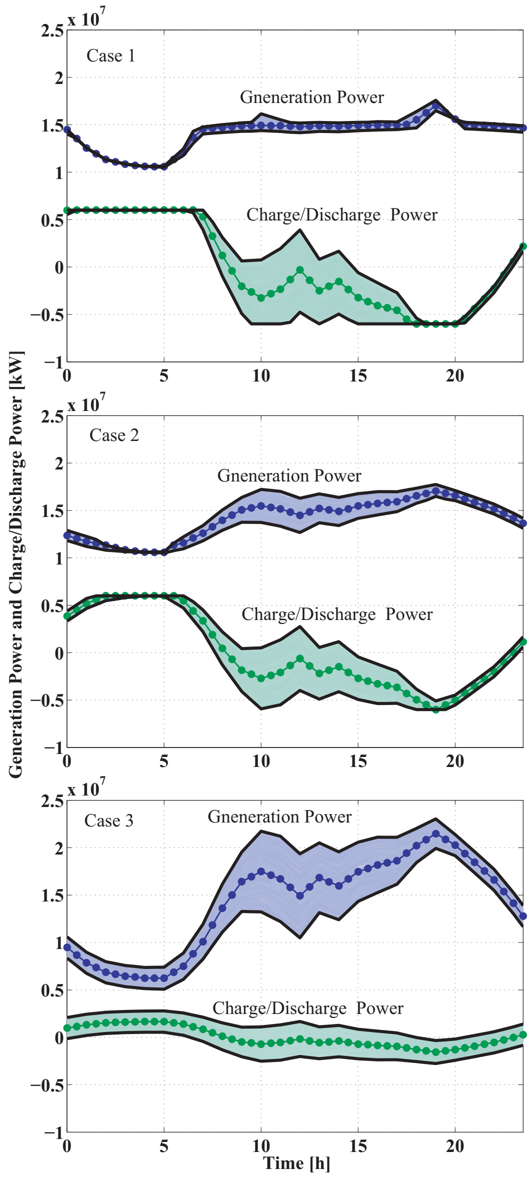

Fig. 2. Power Generation Plan and Battery Charge Cycles.

Thus, we have

$$\begin{aligned}\hat{A} &:= P^{\mathsf{T}}A^{-1}P - P^{\mathsf{T}}A^{-1}\Xi A^{-1}P\\ &= P^{\mathsf{T}}XA^{-1}P + P^{\mathsf{T}}XA^{-1}\Xi A^{-1}P\\ &= P^{\mathsf{T}}XA^{-1}YP,\end{aligned}$$

where the last equality comes from $X^2 = X$ and

$$XA^{-1}Y = X^2A^{-1} = XA^{-1}.$$

In addition, from the facts that

$$PP^{\mathsf{T}}X = (I_n - \overline{P}\,\overline{P}^{\mathsf{T}})X = X$$

and

$$YPP^{\mathsf{T}} = Y(I_n - \overline{P}\,\overline{P}^{\mathsf{T}}) = Y,$$

we obtain

$$P\hat{A}P^{\mathsf{T}} = XA^{-1}Y = A^{-1}Y$$

by $XA^{-1} = A^{-1}Y$ and $Y^2 = Y$. Note that $\hat{A} = (P^{\mathsf{T}}AP)^{-1}$ holds as shown in [15], and thus it follows that

$$Y = AP(P^{\mathsf{T}}AP)^{-1}P^{\mathsf{T}}.$$

Hence the claim follows. ■

REFERENCES

[1] L. Jaulin, *Applied interval analysis: with examples in parameter and state estimation, robust control and robotics*. Springer, 2001.

[2] E. Hansen and G. W. Walster, *Global optimization using interval analysis: revised and expanded*. CRC Press, 2003, vol. 264.

[3] Q. Lin and G. Huang, "IPEM: an interval-parameter energy systems planning model," *Energy Sources, Part A*, vol. 30, no. 14-15, pp. 1382–1399, 2008.

[4] Y. Zhu, Y. Li, and G. Huang, "Planning municipal-scale energy systems under functional interval uncertainties," *Renewable Energy*, vol. 39, no. 1, pp. 71–84, 2012.

[5] L. He, G. H. Huang, G. Zeng, and H. Lu, "An interval mixed-integer semi-infinite programming method for municipal solid waste management," *Journal of the Air & Waste Management Association*, vol. 59, no. 2, pp. 236–246, 2009.

[6] J. A. Petersen and M. Bodson, "Constrained quadratic programming techniques for control allocation," *Control Systems Technology, IEEE Transactions on*, vol. 14, no. 1, pp. 91–98, 2006.

[7] T. Dwyer, Y. Koren, and K. Marriott, "Drawing directed graphs using quadratic programming," *Visualization and Computer Graphics, IEEE Transactions on*, vol. 12, no. 4, pp. 536–548, 2006.

[8] G. F. Reid and L. Hasdorff, "Economic dispatch using quadratic programming," *Power Apparatus and Systems, IEEE Transactions on*, no. 6, pp. 2015–2023, 1973.

[9] Y. Nesterov, A. S. Nemirovskii, and Y. Ye, *Interior-point polynomial algorithms in convex programming*. SIAM, 1994, vol. 13.

[10] R. Cottle, J. Pang, and R. Stone, "The linear complementarity problem. 1992," *AP, New York*.

[11] S.-T. Liu and R.-T. Wang, "A numerical solution method to interval quadratic programming," *Applied mathematics and computation*, vol. 189, no. 2, pp. 1274–1281, 2007.

[12] W. Li and X. Tian, "Numerical solution method for general interval quadratic programming," *Applied mathematics and computation*, vol. 202, no. 2, pp. 589–595, 2008.

[13] B. Petiau, "Confidence interval estimation for short-term load forecasting," in *PowerTech, 2009 IEEE Bucharest*. IEEE, 2009, pp. 1–6.

[14] A. Seppälä, *Load research and load estimation in electricity distribution*. Technical Research Centre of Finland, 1996.

[15] T. Ishizaki, H. Sandberg, K. H. Johansson, K. Kashima, J.-i. Imura, and K. Aihara, "Structured model reduction of interconnected linear systems based on singular perturbation," in *American Control Conference (ACC), 2013*. IEEE, 2013, pp. 5544–5549.