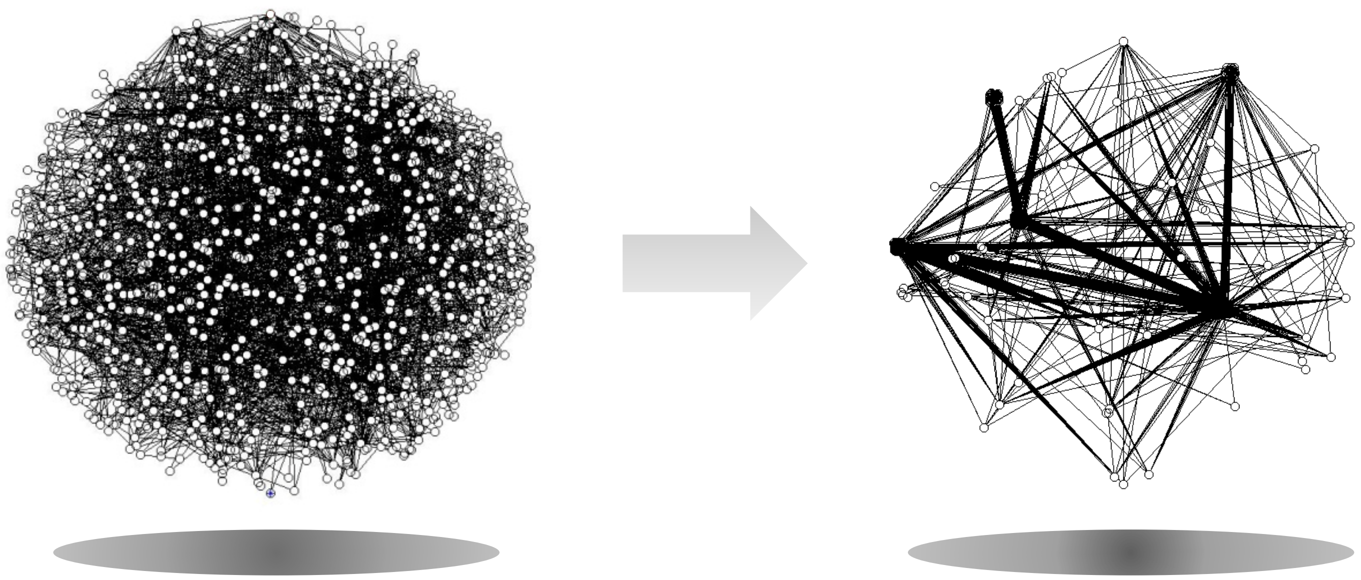
# **Clustered Model Reduction** of Interconnected Second-Order Systems and Its Applications to Power Systems



Takayuki Ishizaki, Jun-ichi Imura (Tokyo Inst. of Tech.)

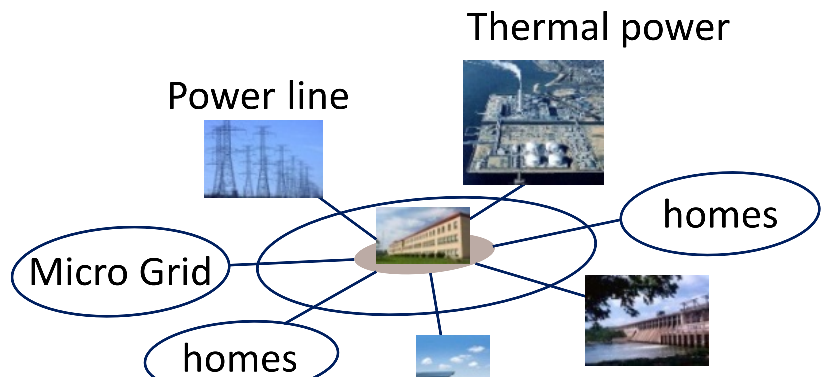
# Outline

- Introduction: Why clustered model reduction?
- Clustered Model Reduction Theory
  - Interconnected first-order systems
  - Extension to second-order networks
- Application to power networks
- Conclusion

# Control of Large-Scale Networks

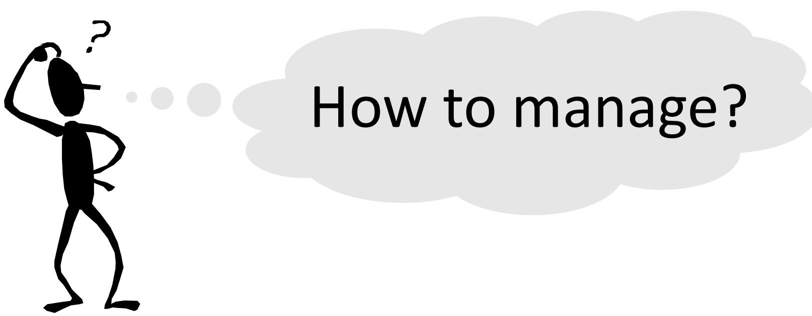
## Power Networks

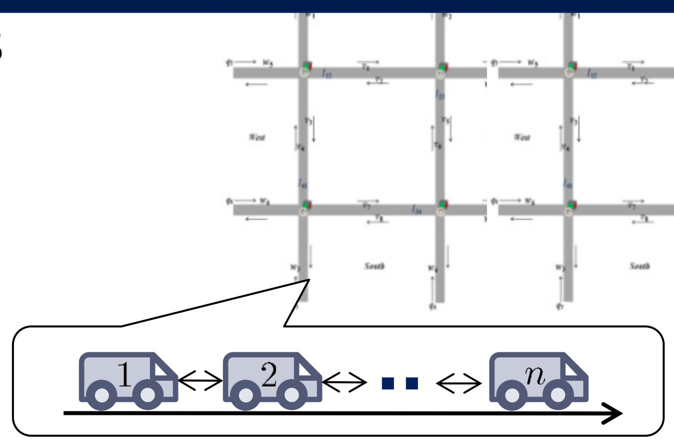
- Tokyo area: 20 million houses
  - Instability may be caused by renewables such as PV, wind




**Model reduction is one prospective approach**

- Center of Tokyo area: 5 million cars
  - Heavy traffic jam



# Standard Model Reduction Framework

✓ $PP^\dagger = I_{\hat{n}}$

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad \xrightarrow[P \in \mathbb{R}^{\hat{n} \times n}]{Px = \hat{x}} \quad \hat{\Sigma} : \begin{cases} \dot{\hat{x}} = PAP^\dagger \hat{x} + PBu \\ \hat{y} = CP^\dagger \hat{x} \end{cases}$$

$x \in \mathbb{R}^n$ $\qquad$ $\hat{x} \in \mathbb{R}^{\hat{n}}, \ \hat{n} < n$

Main goal: Find $P$ such that $\|y - \hat{y}\|$ is small enough

+ stability of error systems, error analysis, low computation cost

Standard methods:

- Balanced truncation, Hankel norm approximation
  - error bound, stability preservation ☺ high computational cost ☹
- Krylov projection
  - lower computation cost ☺ possibly unstable model, no error bound ☹

# Application of Standard Methods to Network Systems

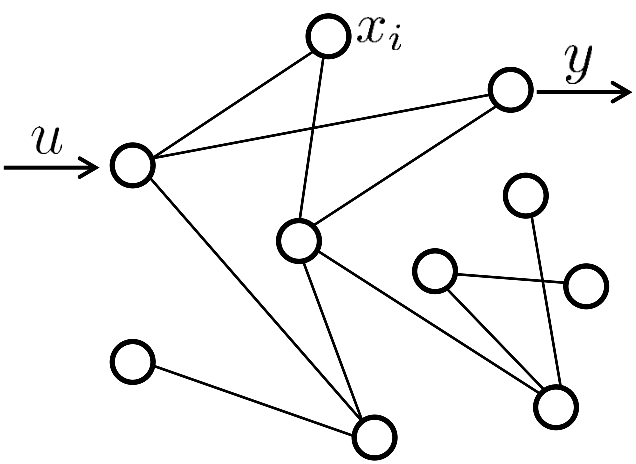
**Drawback: Network structure is lost through reduction**

<u>Network system</u>

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

<u>Reduced model</u>

$$\hat{\Sigma} : \begin{cases} \dot{\hat{x}} = PAP^{\dagger}\hat{x} + PBu \\ \hat{y} = CP^{\dagger}\hat{x} \end{cases}$$

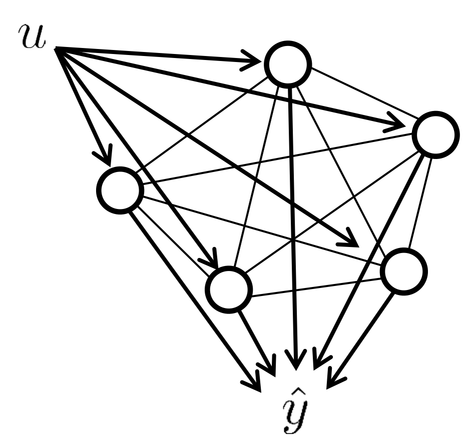



$$Px = \hat{x}$$

$$\longrightarrow$$

$$P \in \mathbb{R}^{\hat{n} \times n}, \ \hat{n} < n$$

Dense matrix



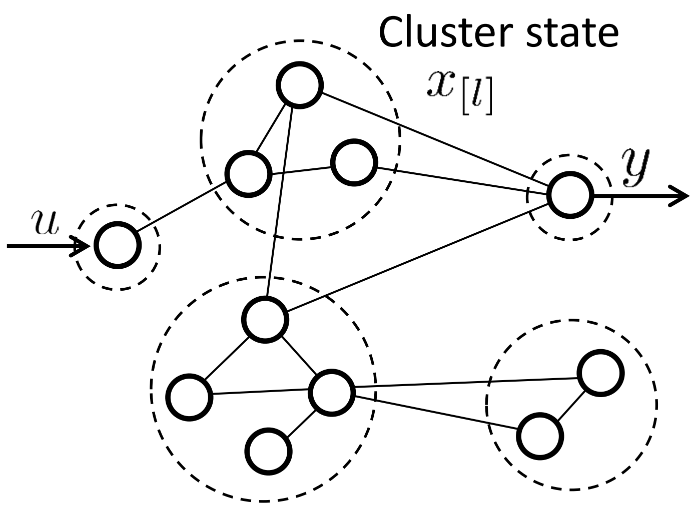

Sparse ☺

Dense ☹

# Clustered Model Reduction

## Network system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$




Sparse ☺

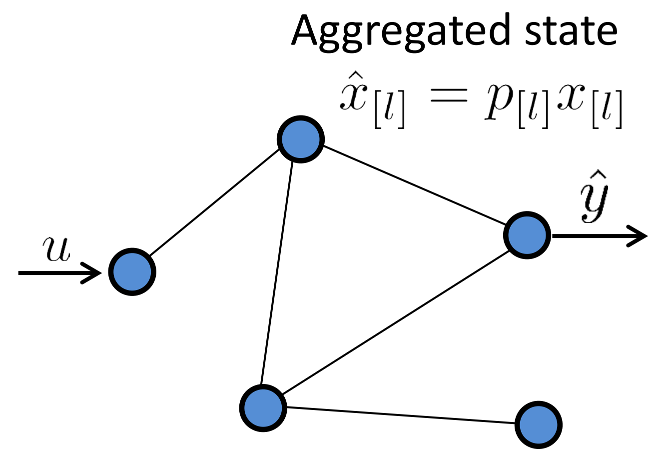
$$Px = \hat{x}$$

$p_{[l]}$ : row vector

$$P = \begin{bmatrix} p_{[1]} & & \\ & p_{[2]} & \\ & & \ddots \end{bmatrix}$$

## Aggregated model

$$\hat{\Sigma} : \begin{cases} \dot{\hat{x}} = PAP^{\dagger}\hat{x} + PBu \\ \hat{y} = CP^{\dagger}\hat{x} \end{cases}$$

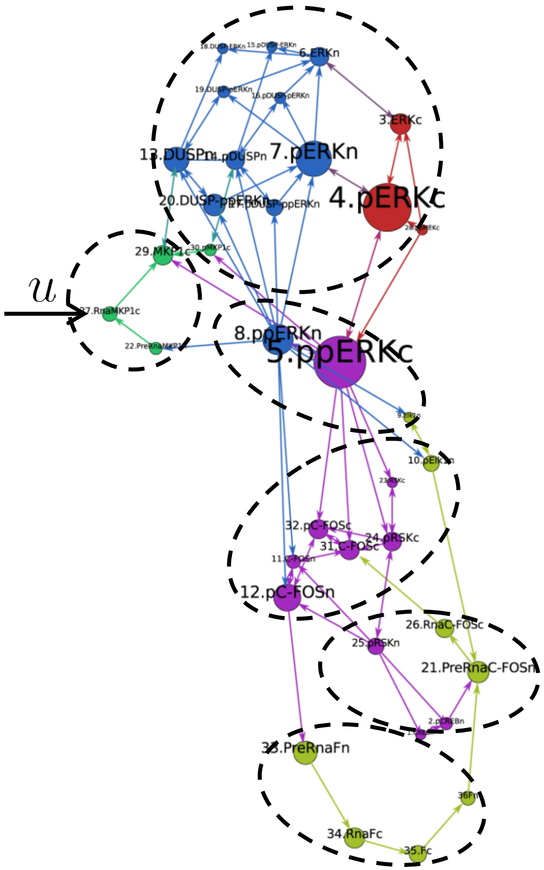



Sparse ☺

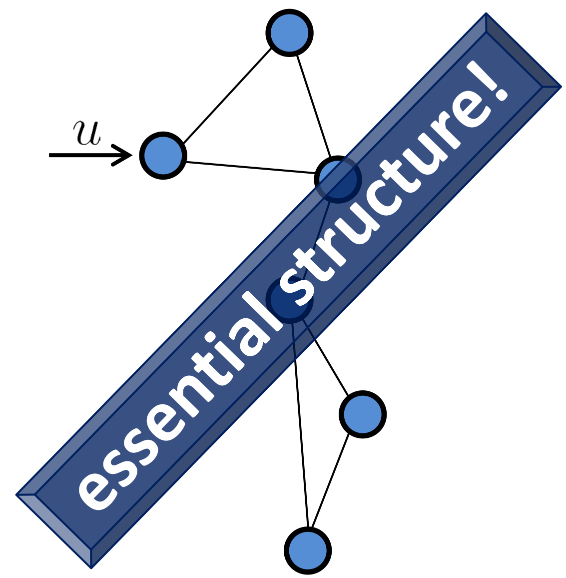
Preservation of network structure among clusters

# Why Clustered Model Reduction?

Gene Network [Mochizuki et al. , J. Theoretical Biology (2010)]




Clustered model reduction




Extract essential structure
to study mechanism of functions

# Outline

- Introduction: Why clustered model reduction?
- Clustered Model Reduction Theory
  - Interconnected first-order systems
  - Extension to second-order networks
- Application to power networks
- Conclusion

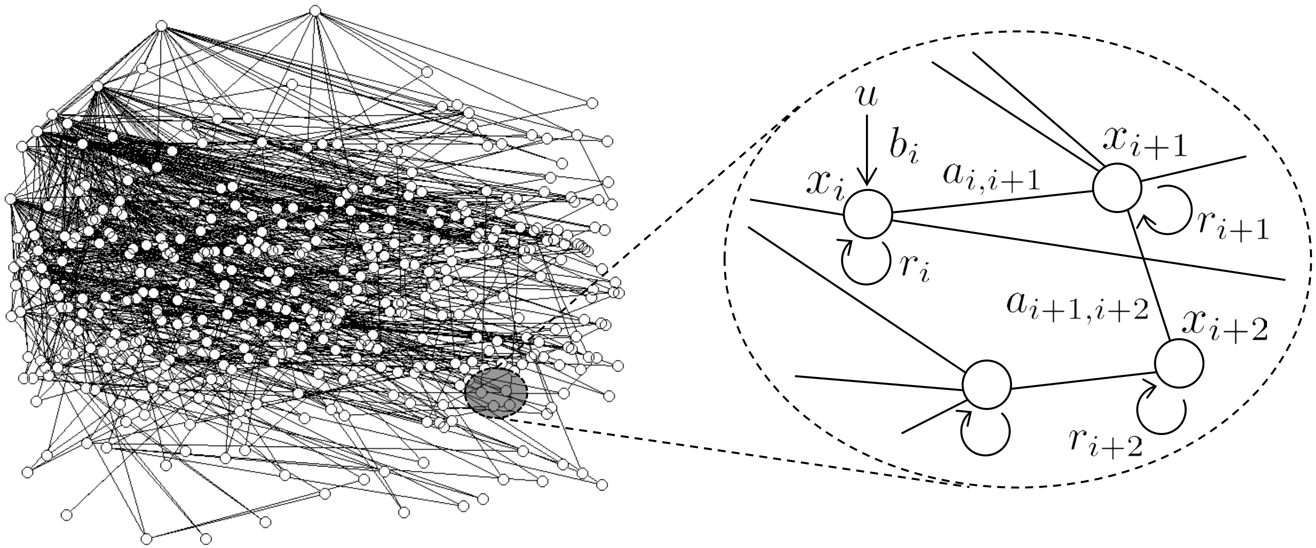
# System Description (First-Order Subsystems)

**[Definition] Bidirectional Network**

$\dot{x} = Ax + Bu$ with $A = \{a_{i,j}\} \in \mathbb{R}^{n \times n}$ and $B = \{b_i\} \in \mathbb{R}^n$

is said to be *bidirectional network* if $A$ is symmetric and stable.

Reaction-diffusion systems: $\dot{x}_i = -r_i x_i + \sum_{j=1, j \neq i}^{n} a_{i,j} (x_j - x_i) + b_i u$



# Clustered Model Reduction Problem

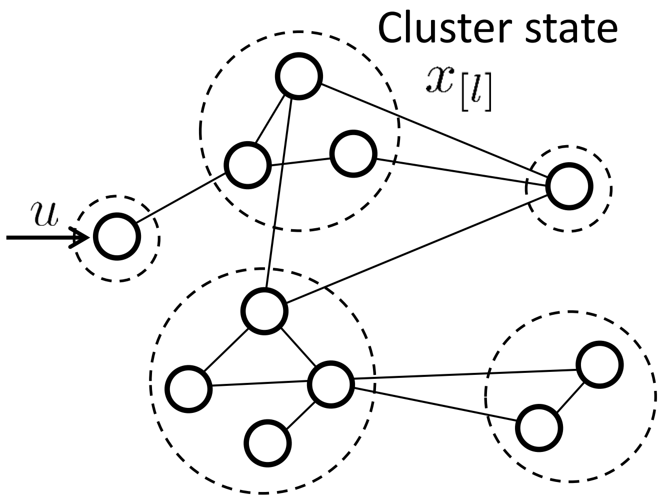
**[Problem]** Given $\epsilon \geq 0$, find a cluster set $\{\mathcal{I}_{[l]}\}_{l \in \mathbb{L}}$ such that

$$\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty} \leq \epsilon$$

where $g(s) := (sI_n - A)^{-1}B$ and $\hat{g}(s) := P^{\mathsf{T}}(sI_{\hat{n}} - PAP^{\mathsf{T}})^{-1}PB.$

## Bidirectional network

$$\Sigma : \dot{x} = Ax + Bu$$




Sparse ☺

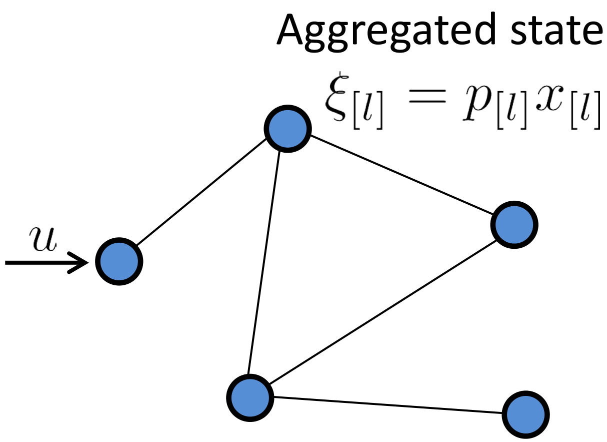
$$Px = \xi$$

$$P = \mathrm{Diag}\left(p_{[1]}, \dots, p_{[\hat{n}]}\right)$$

$$p_{[l]} = \frac{1}{\sqrt{|\mathcal{I}_{[l]}|}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^{\mathsf{T}}$$

## Aggregated model

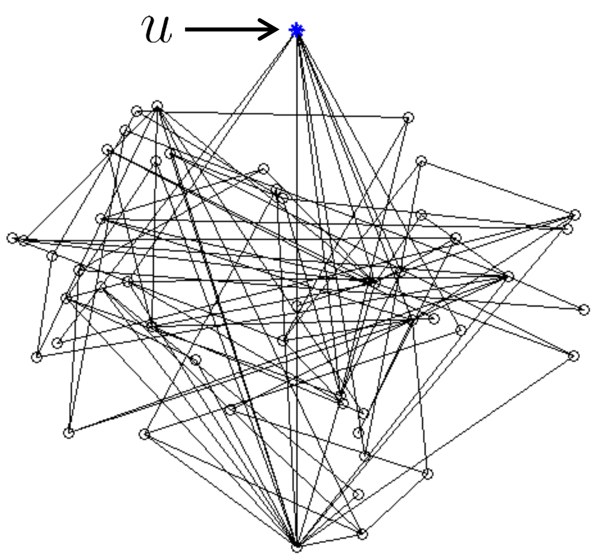
$$\hat{\Sigma} : \begin{cases} \dot{\xi} &= PAP^{\mathsf{T}}\xi + PBu \\ \hat{x} &= P^{\mathsf{T}}\xi \end{cases}$$




Sparse ☺

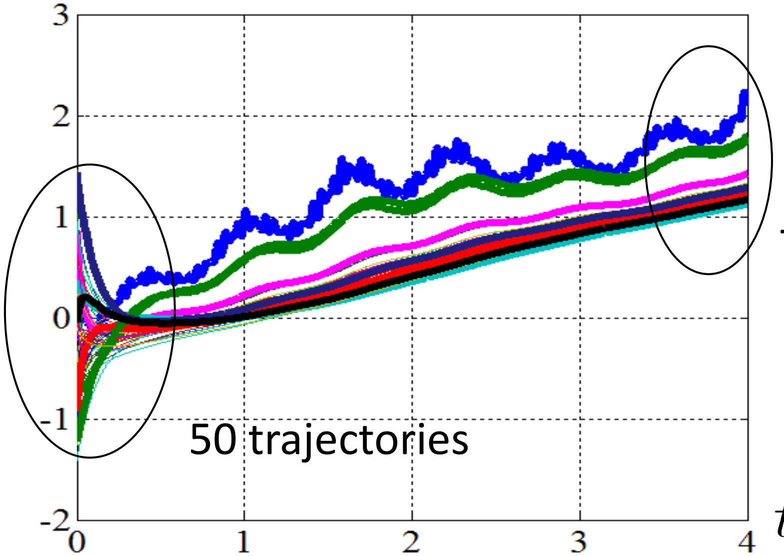
# How to Formulate Reducibility?

<u>Bidirectional network</u> $\dot{x} = Ax + Bu$



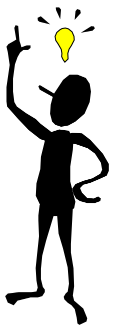

50 nodes, nonzero $a_{i,j}$ is randomly chosen from $(0, 1]$

[State trajectory under random $u$ ]




$x_i(t) - x_j(t) \equiv 0, \quad \forall u(t)$

Local uncontrollability!



**[Definition] Reducible cluster**

A cluster $\mathcal{I}_{[l]}$ is said to be <u>*reducible*</u> if under any input signal $u(t)$

$$x_i(t) \equiv x_j(t), \quad \forall i, j \in \mathcal{I}_{[l]}.$$

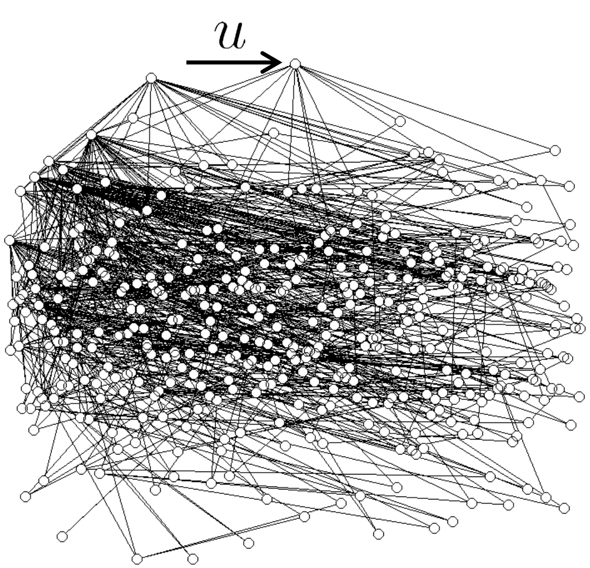
# Positive Tridiagonalization

**[Lemma]** For every bidirectional network $(A, B)$, there exists a unitary $H \in \mathbb{R}^{n \times n}$ such that $(\tilde{A}, \tilde{B}) = (H^\top A H, H^\top B)$ has the following structure.

Bidirectional network $(A, B)$

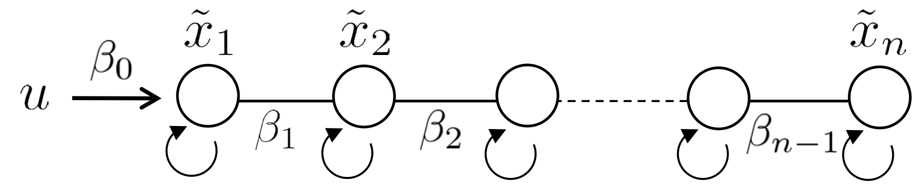
$$\dot{x} = Ax + Bu, \quad A = A^\top$$

(not necessarily positive)




$$x = H\tilde{x}$$

Positive tridiagonal realization $(\tilde{A}, \tilde{B})$



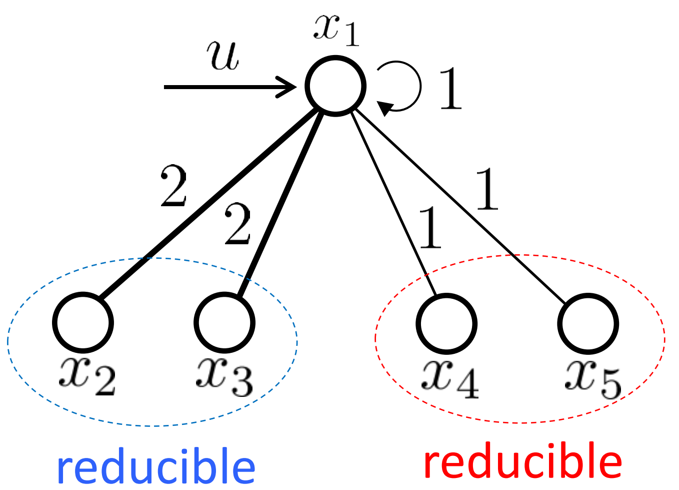

Metzler

$$\tilde{A} = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} \beta_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

✓ $\beta_i \geq 0$

# Reducibility Characterization

Bidirectional network $(A, B)$

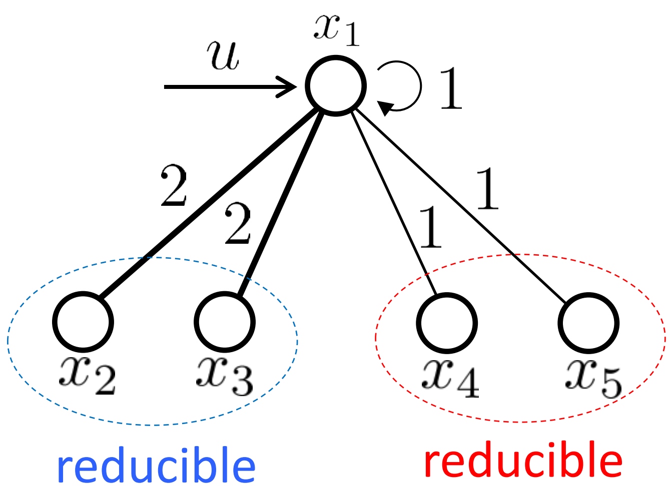



$$A = \begin{bmatrix} -(6+1) & 2 & 2 & 1 & 1 \\ 2 & -2 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

－（graph Laplacian＋diagonal）

# Reducibility Characterization

Bidirectional network $(A, B)$




$\left[\begin{array}{ll}(\tilde{A}, \tilde{B}) & \text{: positive tridiagonal realization} \\ H & \text{: transformation matrix}\end{array}\right.$

<u>Index matrix</u>

$$\Phi := H\mathrm{diag}(-\tilde{A}^{-1}\tilde{B})$$

Characterization in frequency domain

$$\Phi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1.20 & -0.20 & 0 & 0 \\ \hline 0 & 1.20 & -0.20 & 0 & 0 \\ \hline 0 & 0.60 & 0.40 & 0 & 0 \\ \hline 0 & 0.60 & 0.40 & 0 & 0 \end{bmatrix}$$

Rows 2–3: identical

Rows 4–5: identical

Equivalent characterization of cluster reducibility

# $\theta$-Reducible Cluster Aggregation

**[Definition]** **$\theta$-Reducibility of Clusters**

$x_i(t) - x_j(t) \simeq 0, \quad \forall u(t)$

Similar behavior

A cluster $\mathcal{I}_{[l]}$ is said to be <u>*$\theta$-reducible*</u> if

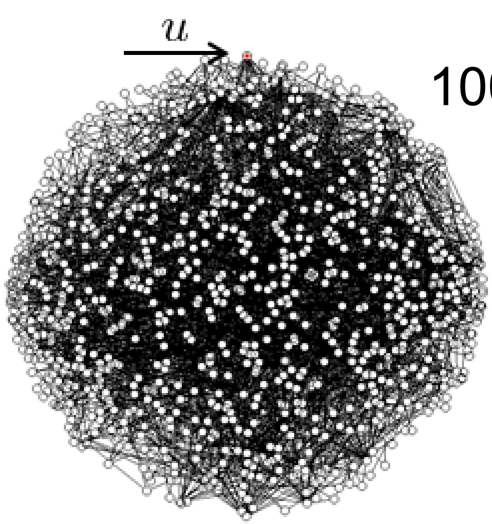
$$\|\mathrm{row}_i[\Phi] - \mathrm{row}_j[\Phi]\|_{l_\infty} \leq \theta, \quad \forall i, j \in \mathcal{I}_{[l]}$$

**[Theorem]** If all clusters are $\theta$-reducible, then

$$\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty} \leq \gamma \sqrt{\textstyle\sum_{l=1}^{\hat{n}} |\mathcal{I}_{[l]}|(|\mathcal{I}_{[l]}| - 1)}\, \theta$$

where $\gamma := \|(PAP^{\mathsf{T}})^{-1}PA\|.$

<u>$\theta$ : coarseness parameter</u>



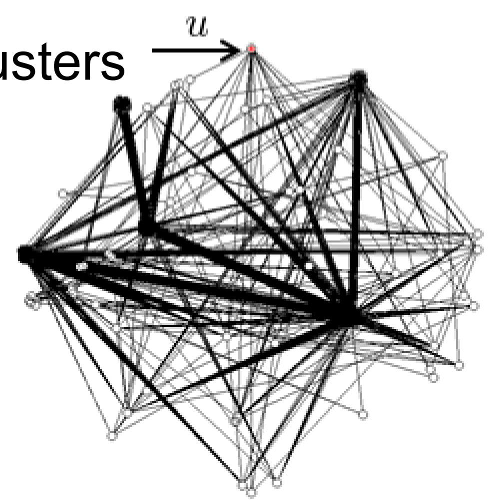

1000 nodes

aggregation

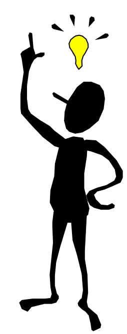
$\theta = 1.5$

about 5% error

47 clusters




Extract essential cluster structure!



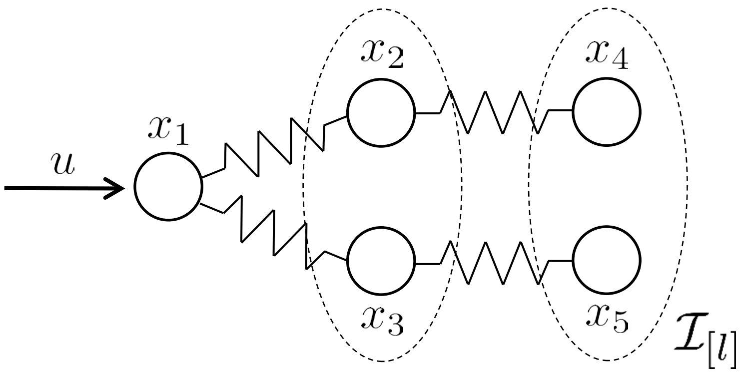
# Outline

- Introduction: Why clustered model reduction?
- Clustered Model Reduction Theory
  - Interconnected first-order systems
  - Extension to second-order networks
- Application to power networks
- Conclusion

# Formulation by Second-Order Systems

**Second-order networks**

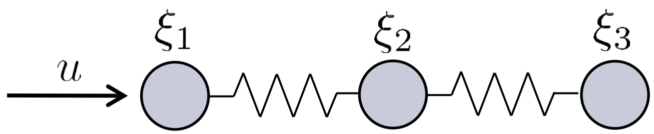
$$\Sigma : \ddot{x} + D\dot{x} + Kx = fu$$




$$Px = \xi$$

**Aggregated model**

$$\hat{\Sigma} : \begin{cases} \ddot{\xi} + PDP^{\mathsf{T}}\dot{\xi} + PKP^{\mathsf{T}}\xi = Pfu \\ \hat{x} = P^{\mathsf{T}}\xi \end{cases}$$




[Problem] Given $\epsilon \geq 0$ , find a cluster set $\{\mathcal{I}_{[l]}\}_{l\in\mathbb{L}}$ such that

$$\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty} \leq \epsilon$$

where $g(s) := (s^2 I_n + sD + K)^{-1} f$ and $\hat{g}(s) := P^{\mathsf{T}}(s^2 I_{\hat{n}} + sPDP^{\mathsf{T}} + PKP^{\mathsf{T}})^{-1} Pf.$

# Extension to Second-Order Networks

First-order representation (2n-dim. system) ✓ $X := \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

$$\Sigma : \begin{cases} \dot{X} = AX + Bu \\ x = CX \end{cases} \quad \text{where} \quad A := \begin{bmatrix} 0 & I_n \\ -K & -D \end{bmatrix} \quad B := \begin{bmatrix} 0 \\ f \end{bmatrix} \quad C := \begin{bmatrix} I_n & 0 \end{bmatrix}$$

Index matrix $\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$ w.r.t. $\begin{cases} \text{position } x \\ \text{velocity } \dot{x} \end{cases}$

**[Definition]** A cluster $\mathcal{I}_{[l]}$ is said to be *$\theta$-reducible* if

$$\|\mathrm{row}_i[\Phi_k] - \mathrm{row}_j[\Phi_k]\|_{l_\infty} \leq \theta, \quad \forall i, j \in \mathcal{I}_{[l]}, \ k \in \{1, 2\}$$

**[Theorem]** If all clusters are $\theta$-reducible, then

$$\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty} \leq \gamma \sqrt{\textstyle\sum_{l=1}^{\hat{n}} |\mathcal{I}_{[l]}|(|\mathcal{I}_{[l]}| - 1)}\, \theta$$

where $\gamma := \sqrt{2} \left\| P(s^2 I_{\hat{n}} + sPDP^{\mathsf{T}} + PKP^{\mathsf{T}})^{-1} [PK \ \ PD] - [I_n \ \ 0] \right\|_{\mathcal{H}_\infty}$.

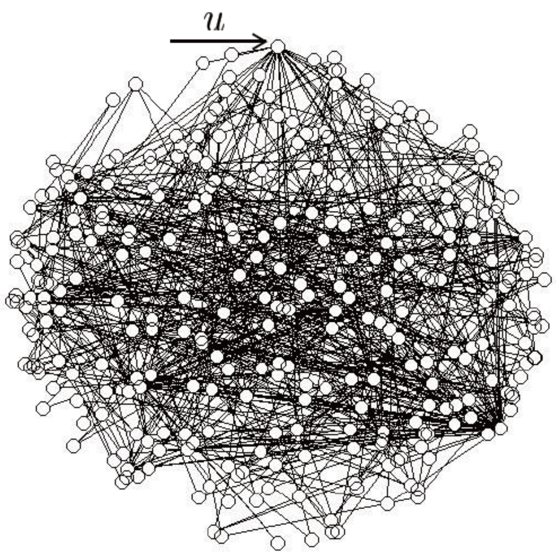
# Outline

- Introduction: Why clustered model reduction?
- Clustered Model Reduction Theory
  - Interconnected first-order systems
  - Extension to second-order networks
- Application to power networks
- Conclusion

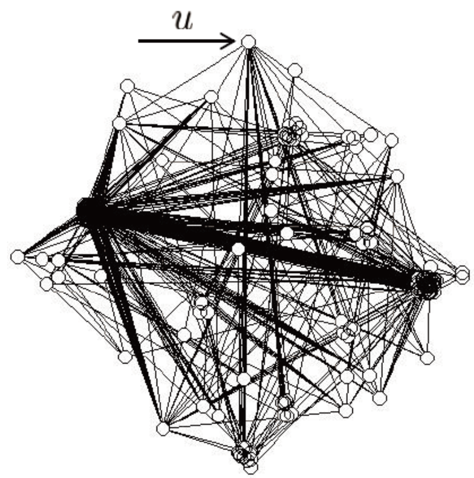
# Numerical Example

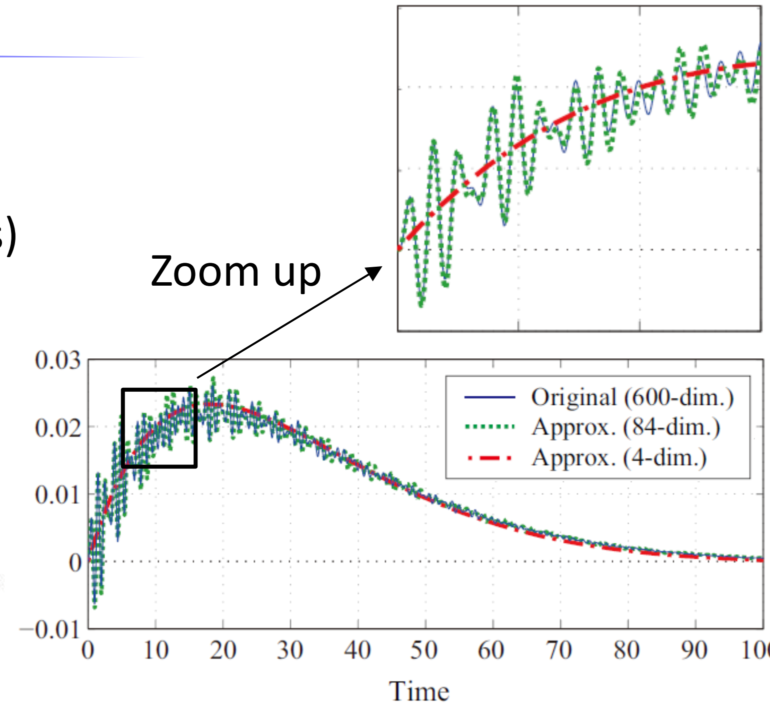
## Power network modeled by swing equation

Original network (300 nodes)

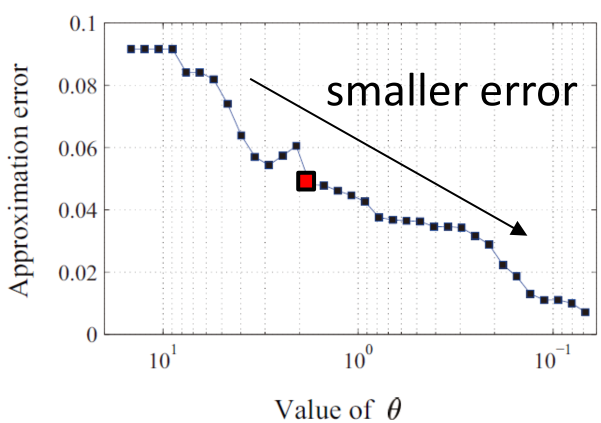


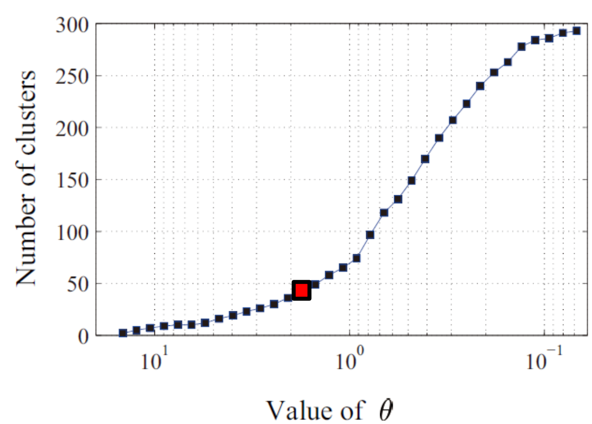

Agg. model (42 clusters)




Position trajectory of a mass




Relative error

$$\frac{\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty}}{\|g(s)\|_{\mathcal{H}_\infty}} = 4.81\%$$

when $\theta = 1.76$

# Concluding Remarks

- <u>Clustered model reduction</u>
  - extract essential information on input-to-state mapping
  - application to power networks model by swing equation
- <u>Future works</u>
  - application to more realistic power networks
  - extension to nonlinear systems
  - application to control system design

[T. Ishizaki et al. IEEE TAC (2014)], [T. Ishizaki et al. NOLTA (2015)], My website, etc.

**Thank you for your attention!**