Multiresolved Control of Discrete-Time Linear Systems Based on Redundant Realization via Wedderburn Rank Reduction

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Abstract— **In this paper, we propose a design method of multiresolved control for discrete-time linear systems. In the proposed control system, we implement a transitory compensator specialized in controlling a short-term system behavior into a standard controller that is designed for controlling a long-term behavior. To establish such control architecture, we construct a low-rank model having the same reachable and observable subspaces as those of the original system in the range of its rank. Then, we derive a redundant state-space realization associated with the low-rank model. A cascaded structure of the redundant realization enables to systematically design a transitory compensator that stabilizes the short-term system behavior while cooperating with a standard controller. The efficiency of the multiresolved control is shown through an illustrative example of frequency control in power networks.**

I. Introduction

In the real world, there can be found a number of systems whose behavior is captured as an interaction among subsystems having different spatiotemporal scales. For example, biological systems [1] are composed of molecules, proteins, cells, and organs that hierarchically interact on multiple spatiotemporal scales. To deal with such nonuniformly distributed systems in the spatiotemporal point of view, it is crucial to figure out and take advantage of essential system properties depending on objectives to be accomplished.

Towards the development of a systematic framework for spatiotemporally multiresolved control [2], this paper proposes a design method of multiple time scale control, in which we explicitly consider the short-term to long-term behavior of systems to be stabilized. To this end, we first construct a low-rank model that can properly capture a short-term system behavior, which can be represented by low-dimensional reachable and observable subspaces. Next, we derive a redundant state-space realization associated with the low-rank model, and then we develop a design method of multiresolved control. By virtue of a cascaded structure of the redundant realization, we can systematically design a transitory compensator that stabilizes the states in possible contingencies, cooperating with a standard controller for normal circumstances.

As the demonstration for the effectiveness of this multiresolved control, we perform a numerical simulation on the stabilization of frequency variations in a power network. In this simulation, we show that a low-dimensional transitory compensator, attached to the average feedback controller, has good ability to stabilize contingency frequency variations arising in a local area.

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To clarify our contribution, we give some references as follows. In [3], a hierarchical control architecture is considered where a low-dimensional approximant is used to construct an additional input signal such that the error between the outputs of the approximant and its original system converges to zero asymptotically. However, the hierarchical control system is practically difficult to implement because it is based on the premise of the possibilities of an exact model reduction, i.e., the low-dimensional approximant can exactly reconstruct the original system behavior, and of the state feedback of the original system.

From the viewpoint of time scale separation, the proposed multiresolved control has a similarity to a control synthesis method based on singular perturbation theory [4]. In this approach, an asymptotic expansion is generally used to analyze the degradation of control performance due to singular perturbation approximation. By contrast, our approach has an advantage that, on the basis of the redundant realization having a tractable cascaded structure, we can analytically manage an approximation error of the low-dimensional model, which corresponds to a long-term system behavior. This redundant realization is different from those used in [5], [6] in the sense that we use state-space expansion to derive a multiresolved state-space representation in a temporal viewpoint, whereas the existing works use it to approximately decouple interconnected systems in a spatial viewpoint.

This paper is organized as follows. In Section II, we derive a low-rank model that can capture the short-term behavior of systems on the basis of matrix decomposition, called Wedderburn rank reduction [7]. Then, in Section III, we propose a synthesis method of multiresolved control based on the redundant realization associated with the low-rank model. Section IV shows an illustrative example for the stabilization of frequency variations in a power network. Finally, concluding remarks are provided in Section V.

Notation: We denote the set of real numbers by $\mathbb{R}$, the n-dimensional identity matrix by I_n, the ith column of I_n by e_i^n, the rank of a matrix M by $\mathrm{rank}(M)$, the image of a matrix M by $\mathrm{im}\,M$, and the kernel of a matrix M by $\ker M$. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *Schur* if

$$\lim_{t\to\infty} x_t = 0, \quad \forall x_0 \in \mathbb{R}^n \tag{1}$$

for the associated recurrence formula $x_{t+1} = Ax_t$. Furthermore, a switching matrix $A_t \in \{A^{(1)}, A^{(2)}\}$ with a fixed switching mode sequence is said to be *transitionally Schur* if (1) holds for $x_{t+1} = A_t x_t$.

II. System Reduction Based on Wedderburn Rank Reduction

A. Wedderburn Rank Reduction

Let $A \in \mathbb{R}^{n\times n}$ be given and denote $\nu := \operatorname{rank}(A)$. For some sequences of vectors $r_i, o_i \in \mathbb{R}^n$, we consider the following biconjugation process associated with A:

$$u_i := r_i - \sum_{j=1}^{i-1} \frac{r_i^\mathsf{T} A v_j}{u_j^\mathsf{T} A v_j} u_j, \quad v_i := o_i - \sum_{j=1}^{i-1} \frac{u_j^\mathsf{T} A o_i}{u_j^\mathsf{T} A v_j} v_j, \quad (2)$$

where $u_1 := r_1$ and $v_1 := o_1$. It has been shown in [7] that $v_i^\mathsf{T} A u_j = 0$ for all $i \neq j$, or equivalently, for

$$\Omega_k := \operatorname{diag}(\omega_1, \ldots, \omega_k), \quad \omega_i := v_i^\mathsf{T} A u_i, \quad (3)$$

it follows that

$$V_k^\mathsf{T} A U_k = \Omega_k, \quad k \in \{1, \ldots, \nu\} \quad (4)$$

where $U_k := [u_1, \ldots, u_k]$ and $V_k := [v_1, \ldots, v_k]$. The biconjugation process in (2) can be regarded as a function that produces a biorthogonal pair satisfying (4). This process is to be denoted by

$$(U_k, V_k) = \mathcal{W}_A(R_k, O_k) \quad (5)$$

where $R_k := [r_1, \ldots, r_k]$ and $O_k := [o_1, \ldots, o_k]$. Note that they satisfy

$$\operatorname{im} U_k = \operatorname{im} R_k, \quad \operatorname{im} V_k = \operatorname{im} O_k \quad (6)$$

for all $k \in \{1, \ldots, \nu\}$.

In this notation, it is also shown that A is decomposed into the sum of rank-one matrices as

$$A = \sum_{i=1}^{\nu} \omega_i^{-1} \phi_i \psi_i^\mathsf{T}, \quad \begin{cases} \phi_i := A u_i \\ \psi_i := A^\mathsf{T} v_i \end{cases} \quad (7)$$

for any R_ν and O_ν, as long as $u_i^\mathsf{T} A v_i \neq 0$ for all $i \in \{1, \ldots, \nu\}$. On the basis of this rank-one decomposition, we can define a matrix having rank k as

$$A_k := \sum_{i=1}^{k} \omega_i^{-1} \phi_i \psi_i^\mathsf{T}, \quad k \in \{1, \ldots, \nu\}, \quad (8)$$

which is called a Wedderburn matrix [7]. This low-rank reduction can be represented in a matrix form as

$$A_k = \Phi_k \Omega_k^{-1} \Psi_k^\mathsf{T}, \quad \Phi_k := A U_k, \quad \Psi_k := A^\mathsf{T} V_k.$$

Then, the following lemma gives a link between the Wedderburn rank reduction and linear systems theory:

Lemma 1: Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, and $C \in \mathbb{R}^{1\times n}$. Consider the biconjugation process in (5) with

$$\begin{aligned} R_k &= [B, AB, \ldots, A^{k-1}B] \\ O_k &= [C^\mathsf{T}, (CA)^\mathsf{T}, \ldots, (CA^{k-1})^\mathsf{T}] \end{aligned} \quad (9)$$

and assume that Ω_k in (3) is nonsingular. Then, for A_k in (8), it follows that

$$A_k^i B = A^i B, \quad C A_k^i = C A^i \quad (10)$$

for all $i \in \{1, \ldots, k\}$.

Let (A, B, C) denote the system matrices of

$$\begin{cases} x_{t+1} = A x_t + B u_t \\ y_t = C x_t. \end{cases} \quad (11)$$

Lemma 1 shows that, by giving the original bases R_k and O_k as in (9), the biconjugation process in (5) produces a system matrix A_k such that the low-rank model (A_k, B, C) has the same $(k+1)$-dimensional reachable and observable subspaces as those of (A, B, C).

B. Relation to the Krylov Projection

Let us consider the following biconjugation process in (5) associated with I_n:

$$(P_k, Q_k) = \mathcal{W}_{I_n}(R_k, O_k) \quad (12)$$

where R_k and O_k are defined as in (9). From the relations in (4) and (6), we see that

$$\operatorname{im} P_k = \operatorname{im} R_k, \quad \operatorname{im} Q_k = \operatorname{im} O_k, \quad Q_k^\mathsf{T} P_k = \Lambda_k$$

where Λ_k is a diagonal matrix. In model reduction theory, the biorthogonal projection of linear systems onto their reachable and observable subspaces is called the Krylov projection. This model reduction technique can produce an approximate model that preserves the first $2k$ Markov parameters of the original system [8], similarly to (10). The Krylov projection model of (A, B, C) is given as $(P_k^\dagger A P_k, P_k^\dagger B, C P_k)$ where

$$P_k^\dagger := \Lambda_k^{-1} Q_k^\mathsf{T}, \quad (13)$$

which satisfies $P_k^\dagger P_k = I_k$.

To show a relation between the Krylov projection and the low-rank model in Section II-A, we use the following facts [7]. There exists an upper bidiagonal matrix $M_k \in \mathbb{R}^{k\times k}$, whose diagonal elements are all one and superdiagonal elements are all nonzero, such that

$$P_k = U_k M_k, \quad Q_k = V_k M_k. \quad (14)$$

In addition, for Ω_k in (3), it follows that

$$Q_k^\mathsf{T} A_k P_k = M_k^\mathsf{T} \Omega_k M_k, \quad (15)$$

which is a symmetric tridiagonal matrix. In this notation, the following fact is proven:

Lemma 2: Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, and $C \in \mathbb{R}^{1\times n}$ be such that (A, B) is reachable and (A, C) is observable. Consider the biconjugation processes in (5) and (12) with R_k and O_k in (9). Then, (16) holds for A_k in (8), where $\beta_0 := CB$ and β_i denotes the ith superdiagonal element of M_k such that (14). Furthermore, define

$$\hat{A}_k := P_k^\dagger A_{k-1} P_k, \quad \hat{B}_k := P_k^\dagger B, \quad \hat{C}_k := C P_k \quad (17)$$

$$Q_k^\mathsf{T} A_{k-1} P_k = \begin{bmatrix} \omega_1 & \omega_1\beta_1 & & & \\ \omega_1\beta_1 & \omega_2 + \omega_1\beta_1^2 & \omega_2\beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \omega_{k-1}\beta_{k-1} \\ & & & \omega_{k-1}\beta_{k-1} & \omega_{k-1}\beta_{k-1}^2 \end{bmatrix}, \quad Q_k^\mathsf{T} B = (CP_k)^\mathsf{T} = \begin{bmatrix} \beta_0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \tag{16}$$

where $P_k^\dagger$ is defined as in (13). Then, for any $R \in \mathbb{R}^{n\times m}$ and $S \in \mathbb{R}^{p\times n}$, it follows that

$$\begin{aligned} S(zI_n - A_{k-1})^{-1}B &= \hat{S}(zI_k - \hat{A}_k)^{-1}\hat{B}_k \\ C(zI_n - A_{k-1})^{-1}R &= \hat{C}_k(zI_k - \hat{A}_k)^{-1}\hat{R} \end{aligned} \tag{18}$$

for all $k \in \{2,\dots,\nu\}$, where $\hat{R} := P_k^\dagger R$ and $\hat{S} := SP_k$. In particular, if (A_{k-1}, R) and (A_{k-1}, S) are reachable and observable, respectively, then $(\hat{A}_k, \hat{R}, \hat{C}_k)$ and $(\hat{A}_k, \hat{B}_k, \hat{S})$ are minimal realizations.

Lemma 2 shows that the Krylov projection of the low-rank model (A_{k-1}, B, C) onto the k-dimensional reachable and observable subspaces of (A, B, C) leads to the exact dimension reduction. This can be regarded as the extraction of a minimal realization, i.e., a reachable and observable realization, from the low-rank model.

III. Multiresolved Control

A. Observation and Stabilization via Redundant Realization

Consider a discrete-time linear system

$$\Sigma : \begin{cases} x_{t+1} = Ax_t + B\boldsymbol{u}_t + R\boldsymbol{w}_t \\ \boldsymbol{y}_t = Cx_t \\ \boldsymbol{z}_t = Sx_t \end{cases} \tag{19}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^{1\times n}$, $R \in \mathbb{R}^{n\times m}$, $S \in \mathbb{R}^{p\times n}$, and $x_0 \in \mathbb{R}^n$. In the rest of this paper, we assume that (A, B) is reachable, (A, R) is stabilizable, (A, C) is observable, and (A, S) is detectable. For convenience of arguments, we express measurable (available) quantities by the bold face, e.g., $\boldsymbol{u}_t$ and $\boldsymbol{y}_t$ in (19).

In the following, we design a compensator that aims at stabilizing the transitory behavior of systems while cooperating with a standard controller. More specifically, we consider a situation where a transitory compensator begins to work at $t = 0$ for stabilizing an unknown disturbance, described as an unknown initial state. Furthermore, we suppose that a standard controller has observed the system state for $t < 0$ to accomplish a control objective in normal circumstances.

We first introduce the following redundant state-space realization, similar to one used in [9]:

Lemma 3: Consider Σ in (19). Let $\tilde{\Sigma}$ be given by (23) where $\hat{\boldsymbol{x}}_0$ is a constant, A_k is defined as in (8) and

$$\Gamma_k := \sum_{i=k}^{\nu} \omega_i^{-1}\phi_i\psi_i^\mathsf{T}. \tag{21}$$

Then, it follows that

$$x_t = x_t^{(1)} + x_t^{(2)}, \quad \forall t \in \{0,1,\dots\} \tag{22}$$

and

$$\begin{cases} \boldsymbol{y}_t = y_t^{(1)} + y_t^{(2)} \\ \boldsymbol{z}_t = z_t^{(1)} + z_t^{(2)}, \end{cases} \quad \forall t \in \{0,1,\dots\} \tag{23}$$

for any $x_0 \in \mathbb{R}^n$ and any sequences of $\boldsymbol{u}_t \in \mathbb{R}$ and $\boldsymbol{w}_t \in \mathbb{R}^m$.

The expanded system $\tilde{\Sigma}$ in (23) can be regarded as a redundant state-space realization of Σ in (19). The constant $\hat{\boldsymbol{x}}_0$ is to be used as a guess of the initial state x_0 available from past observation, i.e., the state observation by a standard controller for $t < 0$. In other words, $x_0 - \hat{\boldsymbol{x}}_0$ in (23) corresponds to an unknown disturbance arising at $t = 0$.

The cascaded structure of $\tilde{\Sigma}$ provides a possibility to use the input signals $\boldsymbol{u}_t$ and $\boldsymbol{w}_t$ for controlling $x_t^{(1)}$ and $x_t^{(2)}$ in an individual manner. Note that $x_t^{(1)}$ has an ability to capture the short-term behavior of Σ owing to the property of the low-rank model shown in Lemma 1. Accordingly, $x_t^{(2)}$ represents the remaining behavior in the sense of the superposition as in (22).

As shown in the relation of (23), even though $\boldsymbol{y}_t$ and $\boldsymbol{z}_t$ are available as the measurement signals, the outputs $y_t^{(i)}$ and $z_t^{(i)}$ of the redundant realization $\tilde{\Sigma}$ cannot be measured as individual signals from the system Σ. To manage this difficulty, we use the following fact, which stems from the matching of the observable subspace shown in Lemma 1:

Lemma 4: Consider $\tilde{\Sigma}$ in (23) and let $\tilde{\Sigma}_{\mathrm{obs}}$ be given by (27), where the sequences of $\boldsymbol{h}_t^{(1)}$ and $\boldsymbol{h}_t^{(2)}$ denote external inputs. If $\boldsymbol{h}_t^{(2)} = 0$ for all $t \in \{0,\dots,k-1\}$, then

$$y_t^{(2)} = \hat{\boldsymbol{y}}_t^{(2)}, \quad \forall t \in \{0,\dots,k-1\} \tag{25}$$

for any $x_0 \in \mathbb{R}^n$ and any sequences of $\boldsymbol{u}_t \in \mathbb{R}$, $\boldsymbol{w}_t \in \mathbb{R}^m$, and $\boldsymbol{h}_t^{(1)} \in \mathbb{R}^n$.

Lemma 4 shows that, as long as $\boldsymbol{h}_t^{(2)} = 0$, the virtual output $y_t^{(2)}$ of the redundant realization $\tilde{\Sigma}$ can be exactly constructed as the output $\hat{\boldsymbol{y}}_t^{(2)}$ of its Luenberger-type observer $\tilde{\Sigma}_{\mathrm{obs}}$ during the limited interval of $t \in \{0,\dots,k-1\}$. From this fact in conjunction with (23), we see that $y_t^{(1)}$ is measurable as $\boldsymbol{y}_t - \hat{\boldsymbol{y}}_t^{(2)}$ during the limited interval. This idea leads to the following result:

Theorem 1: Consider $\tilde{\Sigma}$ in (23) and $\tilde{\Sigma}_{\mathrm{obs}}$ in (27). Let $\boldsymbol{h}_t^{(1)}, \boldsymbol{h}_t^{(2)} \in \mathbb{R}^n$ in (27) be given by

$$\begin{aligned} \boldsymbol{h}_t^{(1)} &= \bar{\delta}_t H^{(1)}\left(\boldsymbol{y}_t - \hat{\boldsymbol{y}}_t^{(1)} - \hat{\boldsymbol{y}}_t^{(2)}\right) \\ \boldsymbol{h}_t^{(2)} &= \delta_t H^{(2)}\left(\boldsymbol{z}_t - \hat{\boldsymbol{z}}_t^{(1)} - \hat{\boldsymbol{z}}_t^{(2)}\right), \end{aligned} \tag{26}$$

where $\boldsymbol{y}_t \in \mathbb{R}$ and $\boldsymbol{z}_t \in \mathbb{R}^p$ associated with Σ in (19) satisfy (23), and

$$\delta_t := \begin{cases} 0, & t \in \{0,\dots,k-1\} \\ 1, & t \in \{k, k+1\dots,\}, \end{cases} \qquad \bar{\delta}_t := 1 - \delta_t. \tag{27}$$

$$\tilde{\Sigma} : \begin{cases} \begin{bmatrix} x_{t+1}^{(2)} \\ x_{t+1}^{(1)} \end{bmatrix} = \begin{bmatrix} A & \Gamma_k \\ 0 & A_{k-1} \end{bmatrix} \begin{bmatrix} x_t^{(2)} \\ x_t^{(1)} \end{bmatrix} + \begin{bmatrix} R\boldsymbol{w}_t \\ B\boldsymbol{u}_t \end{bmatrix}, \qquad \begin{bmatrix} x_0^{(2)} \\ x_0^{(1)} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{x}}_0 \\ x_0 - \hat{\boldsymbol{x}}_0 \end{bmatrix} \\ \begin{bmatrix} y_t^{(2)} \\ y_t^{(1)} \end{bmatrix} = \begin{bmatrix} Cx_t^{(2)} \\ Cx_t^{(1)} \end{bmatrix} \\ \begin{bmatrix} z_t^{(2)} \\ z_t^{(1)} \end{bmatrix} = \begin{bmatrix} Sx_t^{(2)} \\ Sx_t^{(1)} \end{bmatrix} \end{cases} \tag{23}$$

$$\tilde{\Sigma}_{\mathrm{obs}} : \begin{cases} \begin{bmatrix} \hat{\boldsymbol{x}}_{t+1}^{(2)} \\ \hat{\boldsymbol{x}}_{t+1}^{(1)} \end{bmatrix} = \begin{bmatrix} A & \Gamma_k \\ 0 & A_{k-1} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{x}}_t^{(2)} \\ \hat{\boldsymbol{x}}_t^{(1)} \end{bmatrix} + \begin{bmatrix} R\boldsymbol{w}_t \\ B\boldsymbol{u}_t \end{bmatrix} + \begin{bmatrix} \boldsymbol{h}_t^{(2)} \\ \boldsymbol{h}_t^{(1)} \end{bmatrix}, \qquad \begin{bmatrix} \hat{\boldsymbol{x}}_0^{(2)} \\ \hat{\boldsymbol{x}}_0^{(1)} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{x}}_0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \hat{\boldsymbol{y}}_t^{(2)} \\ \hat{\boldsymbol{y}}_t^{(1)} \end{bmatrix} = \begin{bmatrix} C\hat{\boldsymbol{x}}_t^{(2)} \\ C\hat{\boldsymbol{x}}_t^{(1)} \end{bmatrix} \\ \begin{bmatrix} \hat{\boldsymbol{z}}_t^{(2)} \\ \hat{\boldsymbol{z}}_t^{(1)} \end{bmatrix} = \begin{bmatrix} S\hat{\boldsymbol{x}}_t^{(2)} \\ S\hat{\boldsymbol{x}}_t^{(1)} \end{bmatrix} \end{cases} \tag{27}$$

Then, $\epsilon_t^{(1)} := x_t^{(1)} - \hat{\boldsymbol{x}}_t^{(1)}$ and $\epsilon_t^{(2)} := x_t^{(2)} - \hat{\boldsymbol{x}}_t^{(2)}$ obey

$$\begin{cases} \epsilon_{t+1}^{(2)} = (A - \delta_t H^{(2)} S)\epsilon_t^{(2)} + (\Gamma_k - \delta_t H^{(2)} S)\epsilon_t^{(1)} \\ \epsilon_{t+1}^{(1)} = (A_{k-1} - \overline{\delta}_t H^{(1)} C)\epsilon_t^{(1)} \end{cases} \tag{28}$$

for any sequences of $\boldsymbol{u}_t \in \mathbb{R}$ and $\boldsymbol{w}_t \in \mathbb{R}^m$. Furthermore, if $A_{k-1} - \overline{\delta}_t H^{(1)} C$ is transitionally Schur and $A - H^{(2)} S$ is Schur, then

$$\lim_{t\to\infty} \epsilon_t^{(1)} = 0, \quad \lim_{t\to\infty} \epsilon_t^{(2)} = 0$$

for any $x_0 \in \mathbb{R}^n$.

Theorem 1 shows that, by switching the output feedback as in (26), the observer $\tilde{\Sigma}_{\mathrm{obs}}$ can estimate the states of the redundant realization $\tilde{\Sigma}$ in an individual manner. The feedback of estimated states leads to a dynamical stabilizing controller as follows:

Theorem 2: In the same notation as that in Theorem 1, assume that $A_{k-1} - \overline{\delta}_t H^{(1)} C$ is transitionally Schur and $A - H^{(2)} S$ is Schur. Let

$$\boldsymbol{u}_t = F^{(1)} \hat{\boldsymbol{x}}_t^{(1)}, \quad \boldsymbol{w}_t = F^{(2)} \hat{\boldsymbol{x}}_t^{(2)}. \tag{29}$$

For $\tilde{\Sigma}$ in (23) satisfying (22) with respect to Σ in (19), if $A_{k-1} + BF^{(1)}$ and $A + RF^{(2)}$ are Schur, then

$$\lim_{t\to\infty} x_t^{(1)} = 0, \quad \lim_{t\to\infty} x_t^{(2)} = 0 \tag{30}$$

for any $x_0 \in \mathbb{R}^n$.

The switching control system in Theorem 2, which we call a multiresolved control system, consists of the cascaded connection of a low-rank component and a full-rank component (i.e., low and high resolution components), each of which stabilizes the states $x_t^{(1)}$ and $x_t^{(2)}$ of the redundant realization $\tilde{\Sigma}$. Clearly, (30) implies the stability of the original x_t in the closed-loop system.

B. Implementation in Minimal Dimension

In this subsection, we consider implementing the low-rank component in the multiresolved control system by a lower-dimensional realization.

Lemma 5: Consider $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, and $C \in \mathbb{R}^{1\times n}$. For A_{k-1} defined as in (8), there exist F_k and H_k such that $A_{k-1} + BF_k$ is Schur, $A_{k-1} - \overline{\delta}_t H_k C$ is transitionally Schur, and

$$\operatorname{im} F_k^{\mathsf{T}} \subset \operatorname{im} O_k, \quad \operatorname{im} H_k \subset \operatorname{im} R_k, \tag{31}$$

where R_k and O_k are defined as in (9).

On the basis of this fact in conjunction with the extraction of the minimal realization in Lemma 2, we obtain a tractable realization of the multiresolved control system as follows:

Theorem 3: Consider Σ in (19). Let

$$K : \begin{cases} \boldsymbol{\xi}_{t+1} = A_K \boldsymbol{\xi}_t + \delta_t H(\boldsymbol{z}_t - \hat{\boldsymbol{z}}_t) + \phi_k \boldsymbol{\gamma}_t, \quad \boldsymbol{\xi}_0 = \hat{\boldsymbol{x}}_0 \\ \boldsymbol{w}_t = F\boldsymbol{\xi}_t \\ \hat{\boldsymbol{y}}_t = C\boldsymbol{\xi}_t \end{cases} \tag{32}$$

where δ_t is defined as in (27) and

$$A_K := A + RF - \delta_t HS.$$

Furthermore, with $\hat{A}_k$, $\hat{B}_k$, and $\hat{C}_k$ in (17), let

$$\pi : \begin{cases} \boldsymbol{\eta}_{t+1} = \hat{A}_\pi \boldsymbol{\eta}_{t+1} + \overline{\delta}_t \hat{H}_k(\boldsymbol{y}_t - \hat{\boldsymbol{y}}_t), \quad \boldsymbol{\eta}_0 = 0 \\ \boldsymbol{u}_t = \hat{F}_k \boldsymbol{\eta}_{t+1} \\ \boldsymbol{\gamma}_t = (e_k^k)^{\mathsf{T}} \boldsymbol{\eta}_{t+1} \\ \hat{\boldsymbol{z}}_t = \hat{S}\boldsymbol{\eta}_{t+1} \end{cases} \tag{33}$$

where $\overline{\delta}_t$ is defined as in (27), $\hat{S} := SP_k$, and

$$\hat{A}_\pi := \hat{A}_k + \hat{B}_k \hat{F}_k - \overline{\delta}_t \hat{H}_k \hat{C}_k.$$

For $\tilde{\Sigma}$ in (23) satisfying (22) with respect to Σ, if $A + RF$, $A - HS$, and $\hat{A}_k + \hat{B}_k \hat{F}_k$ are Schur and $\hat{A}_k - \overline{\delta}_t \hat{H}_k \hat{C}_k$ is transitionally Schur, then (30) holds for any $x_0 \in \mathbb{R}^n$.

Theorem 3 shows that the multiresolved control system can be implemented as the conventional observer-based state

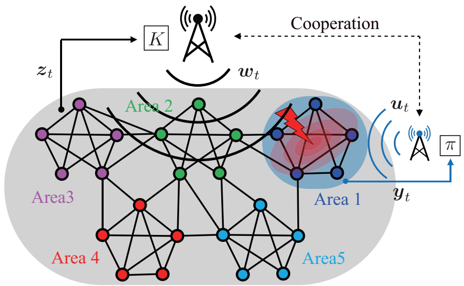

Fig. 1. Interconnection structure of power generators.

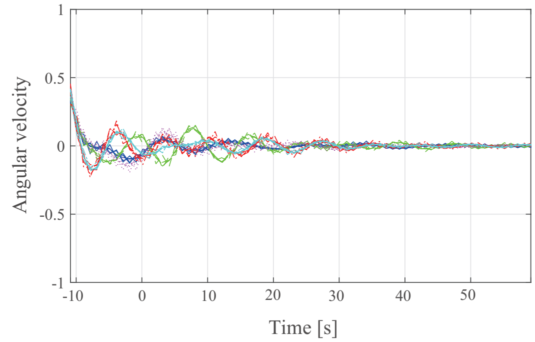

Fig. 2. Average feedback control for synchronous frequency variation.

feedback controller K in (32) to which the low-dimensional compensator π in (33) is attached. The low-dimensional part can be regarded as a transitory compensator that stabilizes the short-term system behavior captured by the low-rank model in Section II.

IV. Illustrative Example

A. Frequency Variation Model for Power Networks

In this section, we perform a numerical simulation on the stabilization of frequency variations. Let us consider a power network consisting of five areas shown in Fig. 1, where the nodes represent power generators and the edges represent the interconnection among the generators. In this network, the generators in each area are densely connected whereas they are sparsely connected among the five areas. Thus, the states of generators belonging to the same area tend to be synchronized each other.

In the following, we model the dynamics of the ith generator as [10]

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i + \sum_{i\neq j} k_{i,j}(\theta_i - \theta_j) = u_i + w_i \tag{34}$$

where m_i denotes the inertial constant, d_i denotes the damping constant, $k_{i,j}$ denotes the coupling strength coefficient between generators, θ_i denotes the angle deviation with respect to a basis generator, and u_i denotes the input torque. The constants are given as follows. As for the coupling strength coefficient between interconnected nodes, we give $k_{i,j} = -1$ if the ith and jth nodes belong to an identical area, and $k_{i,j} = -0.1$ if they belong to different areas. Furthermore

$$k_{i,i} = \begin{cases} 1 - \displaystyle\sum_{i\neq j} k_{i,j}, & i \in \{1, 25\} \\ -\displaystyle\sum_{i\neq j} k_{i,j}, & i \in \{1,\dots,25\}\backslash\{1,25\}. \end{cases}$$

The damping constant is given as $d_i = 0.02$ and the inertial constant is given randomly as $m_i \in [0.35, 0.65]$ for each i.

For the control input w_i, the same signal is supposed to be applied to all the generators as

$$w := w_1 = \cdots = w_{25}. \tag{35}$$

On the other hand, the control input u_i is applied to the generators belonging to the first area as

$$u := u_1 = \cdots = u_5, \quad u_6 = \cdots = u_{25} = 0. \tag{36}$$

Furthermore, the measurement output for constructing the input signal w_i is supposed to be the frequency variation averaged over all generators, denoted by

$$z := \dot{\theta}_1 + \cdots + \dot{\theta}_{25}. \tag{37}$$

Similarly, the measurement output for the input signal u_i is available as the frequency variation averaged over the first area generators, denoted by

$$y := \dot{\theta}_1 + \cdots + \dot{\theta}_5. \tag{38}$$

In this setting, we obtain Σ in (19) by applying the temporal discretization based on the zero-order hold with the sampling period of one second.

B. Simulation Result

1) The Case of Average Feedback Control: First, for the frequency variation model in Section IV-A, we design an average feedback controller that uses $\boldsymbol{w}_t$ and $\boldsymbol{z}_t$, which are the temporally discretized versions of w in (35) and z in (37), as its input and output signals. This controller corresponds to K in (32), for which, on the basis of the LQR design technique, we find the feedback gains F and H such that $A + RF$ and $A - HS$ are Schur.

From the setting of $\boldsymbol{w}_t$ and $\boldsymbol{z}_t$, we can expect that this average feedback controller works well for frequency variations synchronized in all generators. Supposing that such synchronous frequency variations arise as normal circumstances, we calculate the response of the control system with a frequency variation that is almost uniformly distributed over all generators. Furthermore, we set the state of K being identical to that of Σ, which can stem from past observation. The trajectories of the angular velocity (frequency) deviations are shown in Fig. 2, where the color of each set of lines corresponds to that of each area in Fig. 1. From this figure, we see that the frequency variation is properly stabilized by the average feedback controller K as time goes on.

Next, in the same controller setting, we calculate the response of the control system where a local frequency variation is additionally injected into only the first area at $t = 0$. More specifically, we consider the situation that the guess of the initial state satisfies

$$\hat{\boldsymbol{x}}_0 = A\hat{\boldsymbol{x}}_{-1} + R\boldsymbol{w}_{-1}, \quad \boldsymbol{w}_{-1} = F\hat{\boldsymbol{x}}_{-1},$$

where $\hat{\boldsymbol{x}}_{-1} = x_{-1}$ holds, and the initial state of Σ satisfies $x_0 = \hat{\boldsymbol{x}}_0 + \zeta$ where ζ denotes the local frequency variation,

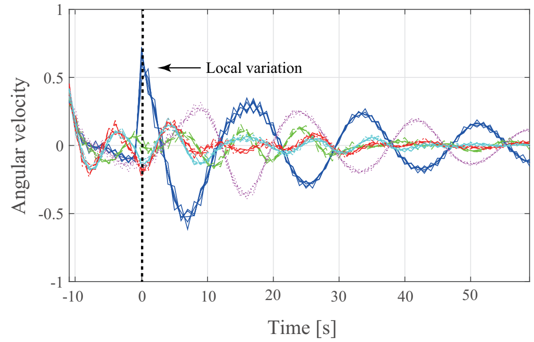

Fig. 3. Average feedback control for asynchronous frequency variation.

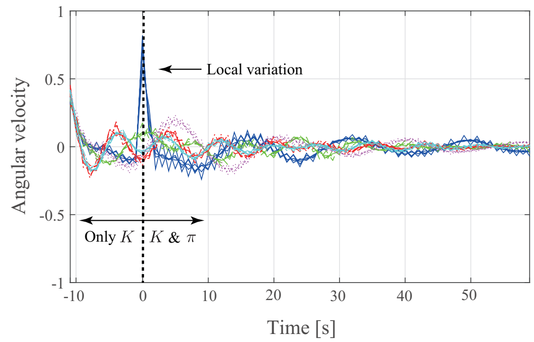

Fig. 4. Multiresolved control for asynchronous frequency variation.

demonstrating a contingency circumstance. From the result shown in Fig. 3, we see that the settling time becomes longer than that in Fig. 2. This result implies that the average feedback controller does not work well for frequency variations arising in a specific area whereas it is effective for uniform variations.

2) The Case of Multiresolved Control: In the following, we show that local frequency variations can be effectively stabilized by implementing a low-dimensional transitory compensator π in (33), which uses $\boldsymbol{u}_t$ and $\boldsymbol{y}_t$, denoting the discretized versions of u in (36) and y in (38), as its input and output signals. Note that, because these signals contain the information on the generators in the first area, they are expected to be suitable for stabilizing local frequency variations arising in the first area.

When $k = 7$, it turns out that A_{k-1} defined as in (8) is Schur. Using this Wedderburn matrix, we construct a seven-dimensional model $(\hat{A}_k, \hat{B}_k, \hat{C}_k)$, for which, on the basis of the LQR design technique, we find the feedback gains $\hat{F}_k$ and $\hat{H}_k$ such that $\hat{A}_k + \hat{B}_k\hat{F}_k$ and $\hat{A}_k - \hat{H}_k\hat{C}_k$ are Schur.

In Fig. 4, we show the system response in the same frequency variation settings as those in Fig. 3 when implementing the seven-dimensional transitory compensator into the average feedback controller in Section IV-B.1. From this figure, where the transitory compensator begins to work at $t = 0$, we see that the stabilization of the frequency variation is well performed in comparison with Fig. 3. Note that this stabilization is achieved by the suitable cooperation of K and π in (32) and (33), demonstrating the efficiency of the proposed multiresolved control. Such cooperation work of different controllers cannot be realized by a *naive* design of each controller because negative interference between controllers possibly occurs if they are implemented at the same time.

In this example, the frequency deviation averaged over the first area generators is fedback to the transitory compensator only for $t \in \{0, \ldots, 7\}$. After $t = 8$, the average deviation over all generators is fedback to the average feedback controller. This switching is represented by δ_t and $\overline{\delta}_t$ in (32) and (33). Note that the transitory compensation fades out as time goes on, and then the multiresolved control coincides with the average feedback control if the state of the transitory compensator converges to zero. On the basis of this control architecture, by designing multiple transitory compensators that are specialized for individual areas in advance, we can realize multiresolved control that can deal with unexpected disturbances arising in a specific area. Such a control strategy is reasonable because contingent disturbances can be dealt with several transitory compensators depending on situations, while a conventional controller can focus on accomplishing a control objective in normal circumstances.

V. Conclusion

In this paper, we have proposed a design method of multiresolved control for discrete-time linear systems. In the multiresolved control, we implement a transitory compensator specialized in stabilizing the short-term system behavior into a conventional controller that is designed for stabilizing the long-term behavior. This control architecture enables efficient handling of possible contingencies, such as unexpected disturbances, besides accomplishing a control objective in normal circumstances. The efficiency of the multiresolved control has been shown through an illustrative example of frequency control in power networks.

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