Clustered Model Reduction of Networked Dissipative Systems

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Abstract— **In this paper, we propose a clustered model reduction method for networked dissipative systems, which consist of identical subsystems having a dissipative property. In the clustered model reduction, we perform both network structure simplification and subsystem approximation, implemented as block-diagonally structured orthogonal projection. To develop this model reduction method, we derive a condition for the existence of block-diagonal Lyapunov functions for networked dissipative systems, given as a matrix inequality that involves interconnection matrices and supply functions. It turns out that the existence of such a structured Lyapunov function is key to guaranteeing stability preservation. In addition, we perform an approximation error analysis in terms of the $\mathcal{H}_2$-norm. The efficiency of the proposed method is demonstrated though numerical experiments for interconnected passive systems.**

I. Introduction

In the real world, there can be found a number of dynamical systems described as a network of subsystems. Examples include power networks, biochemical networks, ecological networks, and so forth [1], [2]. Because such networks are generally modeled as large-scale systems and their network structures are often complex, model reduction techniques for networked systems are expected to be a useful tool to simplify their analyses and syntheses.

In the literature, various model reduction methods can be found; see, e.g. [3] for an overview. In particular, the balanced truncation, the Krylov projection, and the Hankel norm approximation are well known. However, even though such methods can systematically produce a good approximate model of original large-scale systems, they only focus on the approximation of their input-to-output properties; thereby not being concerned with their specific state-space realization. This implies that an interconnection structure in resultant approximate models is not especially taken care of. In this sense, they are not necessarily suitable for the reduction of networked systems.

Against this background, the authors have developed a model reduction method to simplify a network structure among subsystems, called *clustered model reduction*, for networked systems composed of one-dimensional and two-dimensional subsystems [4]–[6]. In this line of works, using a notion of network clustering to find disjoint sets of subsystems to be aggregated, we have derived approximation error bounds in terms of the $\mathcal{H}_2$-norm and the $\mathcal{H}_\infty$-norm. In particular, we clarify in [5] that the existence of diagonal Lyapunov functions is key to guaranteeing stability preservation for the networks of one-dimensional subsystems; stability preservation in [4], [6] can be readily achieved on the premise of the symmetry or second-order form of original systems.

As an extension of these developments, in this paper, we propose a reduction method for networked systems composed of higher-dimensional subsystems. In particular, focusing on a homogeneous network composed of dissipative subsystems, we propose a generalized version of clustered model reduction where we perform both network structure simplification and subsystem approximation. As for stability preservation, we utilize the fact that an interconnected dissipative system admits a block-diagonal Lyapunov function, which can be viewed as a generalization of diagonal ones in the case of one-dimensional subsystems. It turns out that the existence of such a structured Lyapunov function is key to guaranteeing the stability of approximate models. Furthermore, we derive an approximation error bound in terms of the $\mathcal{H}_2$-norm as a natural generalization of that derived in [5]. It should be noted that our stability analysis based on subsystem dissipativity is different from that performed in, e.g. [7], [8], where the eigendecomposition of interconnection matrices is used to decouple the stability analysis of entire networks into those in the subsystem size.

Finally, several references on network structure-preserving model reduction are in order. In [9], a reduction method has been developed for networked systems, where the Krylov projection or the balanced truncation is applied to each subsystem for network structure preservation. However, this method is not concerned with simplifying the network structure among subsystems, but approximating the transfer function of subsystems. Similar approaches are taken also in [10], [11]. On the other hand, reduction methods for simplifying network structures can also be found as [12]–[14]. Similarly to our works, these papers consider aggregating the clusterized nodes of original networks, each of whose node corresponds to a one-dimensional or two-dimensional subsystem. In contrast to our error bound derivation with respect to general clustering, they calculate an explicit value of approximation errors by focusing on a special class of clustering, called equitable partitions [2], and a particular input-to-output mapping. Note that they do not explicitly care about the stability of approximate models because it can be ensured by the Laplacian or port-Hamiltonian structures of original systems, similarly to [4], [6].

The rest of this paper is structured as follows. In Section II, we first perform the stability analysis based on subsystem dissipativity. In Section III, on the basis of the stability anal-

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ysis, we develop a method of clustered model reduction for homogeneous networks composed of dissipative subsystems. For the implementation of the clustered model reduction, we propose a sequential algorithm for finding reducible clusters. Then, in Section IV, we perform a numerical experiment to demonstrate the efficiency of the proposed clustered model reduction method. Finally, concluding remarks are provided in Section V.

Notation: We denote the set of real numbers by $\mathbb{R}$, the cardinality of a set $\mathcal{I}$ by $|\mathcal{I}|$, the identity matrix by I, the ith column of the n-dimensional identity matrix by e_i^n, the trace of a square matrix M by $\mathrm{tr}\, M$, the Kronecker product of A and B by $A \otimes B$, the set of orthogonal projectors by

$$\mathcal{P}^{n\times\hat{n}} := \{P \in \mathbb{R}^{n\times\hat{n}} : P^{\mathsf{T}}P = I,\ \hat{n} \le n\},$$

the orthogonal complement of $P \in \mathcal{P}^{n\times\hat{n}}$ by $\overline{P} \in \mathcal{P}^{n\times(n-\hat{n})}$, and the block diagonal matrix having matrices $M_1, \ldots, M_n$ on its block diagonal by $\mathrm{diag}(M_1, \ldots, M_n)$.

For a set $\mathcal{I} \subseteq \{1, \ldots, n\}$, let $e_{\mathcal{I}}^n \in \mathbb{R}^{n\times|\mathcal{I}|}$ denote the matrix composed of e_i^n for $i \in \mathcal{I}$. The 2-induced norm and the Frobenius norm of a matrix M are defined by

$$\|M\| := \sqrt{\lambda_{\max}(M^{\mathsf{T}}M)}, \quad \|M\|_{\mathrm{F}} := \sqrt{\mathrm{tr}\, M^{\mathsf{T}}M},$$

where $\lambda_{\max}(\cdot)$ denotes the maximal eigenvalue. The positive definiteness and the positive semidefiniteness of a symmetric matrix M are denoted by $M \succ 0$ and $M \succeq 0$. The negative definiteness is denoted similarly. The $\mathcal{H}_\infty$-norm of a stable transfer matrix and the $\mathcal{H}_2$-norm of a stable proper transfer matrix are defined by

$$\begin{aligned}\|G(s)\|_{\mathcal{H}_\infty} &:= \sup_{\omega\in\mathbb{R}} \|G(j\omega)\| \\ \|G(s)\|_{\mathcal{H}_2} &:= \sqrt{\tfrac{1}{2\pi}\textstyle\int_{-\infty}^{\infty} \|G(j\omega)\|_{\mathrm{F}}^2 d\omega},\end{aligned}$$

respectively.

II. Preliminaries of Homogeneous Networks

A. System Description

We consider a network of N identical subsystems, whose dynamics is expressed as

$$\Sigma_i : \begin{cases} \dot{x}_i &= Ax_i + Bu_i \\ y_i &= Cx_i, \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}^n$ denotes the state of the ith subsystem, and $u_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^m$ denote its input and output signals to be used for the interconnection among subsystems. For the stacked input and output signals denoted as

$$\boldsymbol{u} := [u_1^{\mathsf{T}}, \ldots, u_N^{\mathsf{T}}]^{\mathsf{T}}, \quad \boldsymbol{y} := [y_1^{\mathsf{T}}, \ldots, y_N^{\mathsf{T}}]^{\mathsf{T}},$$

the subsystem interconnection is defined by

$$\boldsymbol{u} = (\Gamma \otimes I)\boldsymbol{y} \tag{2}$$

where $\Gamma \in \mathbb{R}^{N\times N}$ denotes an interconnection matrix. Note that this interconnection is rewritten as $u_i = \sum_{j=1}^N \gamma_{i,j} y_j$ where $\gamma_{i,j}$ denotes the (i,j)-element of Γ. Then, for the stacked state denoted as

$$\boldsymbol{x} := [x_1^{\mathsf{T}}, \ldots, x_N^{\mathsf{T}}]^{\mathsf{T}},$$

the entire network system is obtained as

$$\boldsymbol{\Sigma} : \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \tag{3}$$

where the state transition matrix is given by

$$\boldsymbol{A} := I \otimes A + \Gamma \otimes BC. \tag{4}$$

B. Stability Criteria Based on Subsystem Dissipativity

In the literature, a stability analysis of $\boldsymbol{\Sigma}$ in (3) based on the eigendecomposition of Γ can be found [7]. Supposing that the interconnection matrix Γ in (4) is diagonalizable, let T be an eigenvector matrix of Γ. Then, from the similarity transformation of $\boldsymbol{A}$ by $T \otimes I$, it can be proven that

$$\mathrm{spec}(\boldsymbol{A}) = \bigcup_{\lambda\in\mathrm{spec}(\Gamma)} \mathrm{spec}(A + \lambda BC), \tag{5}$$

where $\mathrm{spec}(\cdot)$ denotes the set of eigenvalues. This would be a practical approach because the stability analysis of networked systems can be decomposed into that of small subsystems i.e. $A + \lambda BC$, which involves the eigenvalues of Γ as a parameter. In theoretical physics, the notion of *master stability functions* [8] is defined though this decomposition, to analyze the synchronization of coupled oscillators.

However, such a decomposition approach is not necessarily suitable for network structure-preserving analyses, such as clustered model reduction. This is because the network structure among subsystems is lost though the diagonalization of Γ. In addition, decomposed subsystems may be complex-valued if Γ has complex eigenvalues. In view of this, it would be desirable to devise a stability criterion not relying on such an eigendecomposition.

To derive another stability criterion, let us utilize the following notion of system dissipativity [15]–[17]. A linear system Σ_i in (1) is said to be *dissipative* with respect to a quadratic supply function

$$s(y_i, u_i) := \begin{bmatrix} y_i^{\mathsf{T}} & u_i^{\mathsf{T}} \end{bmatrix} Q \begin{bmatrix} y_i \\ u_i \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{yy} & Q_{yu} \\ Q_{yu}^{\mathsf{T}} & Q_{uu} \end{bmatrix}, \tag{6}$$

where both Q_{yy} and Q_{uu} are assumed to be symmetric, if there exists a quadratic storage function $f(x_i) := x_i^{\mathsf{T}} V x_i$ with $V \succ 0$ such that

$$\dot{f}(x_i) < s(y_i, u_i) \tag{7}$$

along the trajectory of Σ_i. It is known that (7), called a dissipative inequality, can be equivalently expressed as the matrix inequality

$$\begin{bmatrix} A^{\mathsf{T}}V + VA & VB \\ B^{\mathsf{T}}V & 0 \end{bmatrix} - \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^{\mathsf{T}} Q \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \prec 0. \tag{8}$$

In the rest of this paper, we assume without loss of generality that $Q_{uu} \succ 0$; this is a necessary condition for (8). On the basis of this subsystem dissipativity, we derive an existence condition for a structured Lyapunov function to prove the stability of $\boldsymbol{\Sigma}$ in (3) according to [17].

Lemma 1: Let a network system $\mathbf{\Sigma}$ in (3) be given, and assume that there exists $V \succ 0$ such that (8) holds for Q in (6). If there exists $D \succ 0$ being diagonal such that

$$D\otimes Q_{yy}+\Gamma^{\mathsf{T}} D\otimes Q_{yu}^{\mathsf{T}}+D\Gamma\otimes Q_{yu}+\Gamma^{\mathsf{T}} D\Gamma\otimes Q_{uu} \preceq 0, \quad (9)$$

then the positive definite function

$$F(\boldsymbol{x}) := \boldsymbol{x}^{\mathsf{T}}\boldsymbol{V}\boldsymbol{x}, \quad \boldsymbol{V} := D \otimes V \quad (10)$$

is a Lyapunov function to prove the stability of $\mathbf{\Sigma}$.

Proof: For the supply function in (6), we have

$$\dot{F}(\boldsymbol{x}) < \sum_{i=1}^{N} d_i s(y_i, u_i) = \begin{bmatrix} \boldsymbol{y}^{\mathsf{T}} & \boldsymbol{u}^{\mathsf{T}} \end{bmatrix} \boldsymbol{Q}_D \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{u} \end{bmatrix} \quad (11)$$

where d_i denotes the ith diagonal element of D and

$$\boldsymbol{Q}_D := \begin{bmatrix} D \otimes Q_{yy} & D \otimes Q_{yu} \\ D \otimes Q_{yu}^{\mathsf{T}} & D \otimes Q_{uu} \end{bmatrix}.$$

From the relation of (2), we see that the last term in (11) is equal to

$$\boldsymbol{y}^{\mathsf{T}}(D\otimes Q_{yy}+\Gamma^{\mathsf{T}} D\otimes Q_{yu}^{\mathsf{T}}+D\Gamma\otimes Q_{yu}+\Gamma^{\mathsf{T}} D\Gamma\otimes Q_{uu})\boldsymbol{y},$$

which is nonpositive if (9) holds. This proves the claim. ■

Lemma 1 shows an existence condition for the block-diagonal Lyapunov function F in (10) when the interconnected subsystems are dissipative. In particular, the condition in (9) can be reduced when the subsystems are passive or bounded real. To see this, let us first suppose that each of subsystems is passive, i.e.

$$Q_{yy} = Q_{uu} = 0, \quad Q_{yu} = I.$$

In this case, (9) is reduced to

$$\Gamma^{\mathsf{T}} D + D\Gamma \preceq 0,$$

which implies the existence of a diagonal Lyapunov function to prove the stability of Γ. It should be noted that such a diagonal Lyapunov function can be found if Γ is semistable and symmetric, or if it is semistable, irreducible, and Metzler [5]. Next, let us suppose that each of subsystems is bounded real, i.e.

$$Q_{yy} = -I, \quad Q_{uu} = \gamma_0^2 I, \quad Q_{yu} = 0,$$

which is equivalent to the $\mathcal{L}_2$-norm of Σ_i being less than γ_0. Then, (9) is reduced to

$$\|D^{\frac{1}{2}}\Gamma D^{-\frac{1}{2}}\| \le \gamma_0^{-1},$$

which corresponds to the small gain theorem for the interconnection and subsystem gains.

C. Regularization of Networked Dissipative Systems

In this subsection, it turns out that the network of dissipative systems admits a realization having a negative definite property. For convenience, we introduce the following terminology.

Definition 1: A linear system $\mathbf{\Sigma}$ in (3) is said to be a *networked dissipative system* if there exists $V \succ 0$ such that (8) holds for Q in (6) and there exists $D \succ 0$ being diagonal such that (9) holds. In particular, it is said to be *regular* if $\boldsymbol{A} + \boldsymbol{A}^{\mathsf{T}} \prec 0$.

In our previous works [5], [18], [19], it has been found that the regularity in Definition 1 is essential to prove the stability of approximate models obtained by orthogonal projection as well as singular perturbation. The results derived below is reliant on the fact that, owing to the existence of the structured Lyapunov function in (10), any networked dissipative system can be transformed into a regular one without destroying its interconnection structure. This can be seen as follows. Let a Cholesky factor V_{c} be such that $V = V_{\mathrm{c}}^{\mathsf{T}} V_{\mathrm{c}}$. Then, the similarity transformation by $\boldsymbol{V}_{\mathrm{c}} := D^{\frac{1}{2}} \otimes V_{\mathrm{c}}$ yields

$$\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{A}}^{\mathsf{T}} = \boldsymbol{V}_{\mathrm{c}}^{-\mathsf{T}} \left\{ \boldsymbol{V}\boldsymbol{A} + \boldsymbol{A}^{\mathsf{T}}\boldsymbol{V} \right\} \boldsymbol{V}_{\mathrm{c}}^{-1} \prec 0.$$

where $\boldsymbol{V}$ is defined as in (10) and $\tilde{\boldsymbol{A}} := \boldsymbol{V}_{\mathrm{c}}\boldsymbol{A}\boldsymbol{V}_{\mathrm{c}}^{-1}$. Note that the interconnection structure of $\tilde{\boldsymbol{A}}$ is same as that of $\boldsymbol{A}$, in the sense that the Boolean structures of $D^{\frac{1}{2}}\Gamma D^{-\frac{1}{2}}$ and Γ are identical owing to $D^{\frac{1}{2}}$ being diagonal. Thus, any networked dissipative system can be assumed to be regular, i.e. (8) and (9) are satisfied by $V = I$ and $D = I$, without loss of generality.

III. Clustered Model Reduction

A. Problem Formulation

In this section, we address a model reduction problem for networked dissipative systems. To this end, let us introduce external input and output signals for $\mathbf{\Sigma}$ in (3) as

$$\mathbf{\Sigma} : \left\{ \begin{array}{l} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{w} \\ \boldsymbol{z} = \boldsymbol{C}\boldsymbol{x}, \end{array} \right. \quad (12)$$

where $\boldsymbol{A}$ is defined as in (4), and $\boldsymbol{B}$ and $\boldsymbol{C}$ are matrices having compatible dimension. For this system, we consider an approximate model described by

$$\hat{\mathbf{\Sigma}} : \left\{ \begin{array}{l} \dot{\hat{\boldsymbol{x}}} = \boldsymbol{P}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{P}\hat{\boldsymbol{x}} + \boldsymbol{P}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{w} \\ \hat{\boldsymbol{z}} = \boldsymbol{C}\boldsymbol{P}\hat{\boldsymbol{x}}, \end{array} \right. \quad (13)$$

where the orthogonal projector $\boldsymbol{P}$ is structured as

$$\boldsymbol{P} := P \otimes H, \quad P \in \mathcal{P}^{N\times\hat{N}}, \quad H \in \mathcal{P}^{n\times\hat{n}}. \quad (14)$$

By this structured projection, the network system is approximated as

$$\boldsymbol{P}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{P} = I \otimes \hat{A} + \hat{\Gamma} \otimes \hat{B}\hat{C}$$

where the reduced matrices are given by

$$\hat{A} := H^{\mathsf{T}}AH, \quad \hat{B} := H^{\mathsf{T}}B, \quad \hat{C} := CH, \quad \hat{\Gamma} := P^{\mathsf{T}}\Gamma P.$$

This implies that the approximate model $\hat{\mathbf{\Sigma}}$ in (13) is a network of $\hat{N}$ identical subsystems, whose dynamics is

given by replacing A, B, and C in (1) with $\hat{A}$, $\hat{B}$, and $\hat{C}$. Furthermore, the aggregated interconnection matrix is $\hat{\Gamma}$.

It should be noted that, if P is a dense matrix, then so is $\hat{\Gamma}$ in general. This implies that the interconnection structure of the original system is lost through the approximation. To preserve an interconnection structure among subsystems, we impose a particular structure on P, stemming from the notion of network clustering [4], [5].

Definition 2: Let $\mathbb{L} := \{1, \ldots, \hat{N}\}$. The family of an index set $\{\mathcal{I}_{[l]}\}_{l\in\mathbb{L}}$ is referred to as a *cluster set*, each of whose elements is called as a *cluster*, if each element $\mathcal{I}_{[l]}$ is a disjoint subset of $\{1, \ldots, N\}$ and satisfies

$$\bigcup_{l\in\mathbb{L}} \mathcal{I}_{[l]} = \{1, \ldots, N\}.$$

Furthermore, an *aggregation matrix* is defined by

$$P := \mathit{\Pi}\,\mathrm{diag}(p_{[1]}, \ldots, p_{[\hat{N}]}) \tag{15}$$

where $p_{[l]} \in \mathcal{P}^{|\mathcal{I}_{[l]}|}$ and $\mathit{\Pi} := [e^N_{\mathcal{I}_{[1]}}, \ldots, e^N_{\mathcal{I}_{[\hat{N}]}}]$.

Note that the permutation matrix $\mathit{\Pi}$ works as interchanging the indices of subsystems according to a cluster set. Owing to the block-diagonal structure of P, the interconnection structure among clusters is preserved into approximate models. On the basis of this definition, we formulate the following structured model reduction problem.

Problem: Let a networked dissipative system $\boldsymbol{\Sigma}$ in (12) be given, and assume that it is regular. Given a constant $\delta \geq 0$, find an aggregation matrix P and an orthogonal projector H in (14) such that the approximate model $\hat{\boldsymbol{\Sigma}}$ in (13) is a networked dissipative system and it satisfies

$$\|\boldsymbol{G}(s) - \hat{\boldsymbol{G}}(s)\|_{\mathcal{H}_2} \leq \delta \tag{16}$$

where the transfer matrices of $\boldsymbol{\Sigma}$ and $\hat{\boldsymbol{\Sigma}}$ are defined by

$$\begin{aligned} \boldsymbol{G}(s) &:= \boldsymbol{C}(sI - \boldsymbol{A})^{-1}\boldsymbol{B}, \\ \hat{\boldsymbol{G}}(s) &:= \boldsymbol{C}\boldsymbol{P}(sI - \boldsymbol{P}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{P})^{-1}\boldsymbol{P}^{\mathsf{T}}\boldsymbol{B}, \end{aligned} \tag{17}$$

respectively.

B. Analysis of Approximate Models

1) Stability Analysis: Let us first analyze the stability of approximate models. The following result shows that the regularity of the original networked dissipative system is preserved into its approximate model for any choice of $\boldsymbol{P}$.

Theorem 1: If a networked dissipative system $\boldsymbol{\Sigma}$ in (12) is regular, then so is the approximate model $\hat{\boldsymbol{\Sigma}}$ in (13) for any aggregation matrix P and any orthogonal projector H in (14).

Proof: Note that (8) and (9) are satisfied by $V = I$ and $D = I$ owing to the regularity of $\boldsymbol{\Sigma}$. To prove the regularity of $\hat{\boldsymbol{\Sigma}}$, we first show that

$$\begin{bmatrix} \hat{A}^{\mathsf{T}} + \hat{A} & \hat{B} \\ \hat{B}^{\mathsf{T}} & 0 \end{bmatrix} - \begin{bmatrix} \hat{C} & 0 \\ 0 & I \end{bmatrix}^{\mathsf{T}} Q \begin{bmatrix} \hat{C} & 0 \\ 0 & I \end{bmatrix} \prec 0.$$

This inequality follows from multiplying $\mathrm{diag}(H, I)$ by (8) from the right side and its transpose from the left side. Next, we show that

$$I \otimes Q_{yy} + \hat{\Gamma}^{\mathsf{T}} \otimes Q^{\mathsf{T}}_{yu} + \hat{\Gamma} \otimes Q_{yu} + \hat{\Gamma}^{\mathsf{T}}\hat{\Gamma} \otimes Q_{uu} \preceq 0.$$

Multiplying $P \otimes I$ by (9) from the right side and its transpose from the left side, we have

$$I \otimes Q_{yy} + \hat{\Gamma}^{\mathsf{T}} \otimes Q^{\mathsf{T}}_{yu} + \hat{\Gamma} \otimes Q_{yu} + P^{\mathsf{T}}\Gamma^{\mathsf{T}}\Gamma P \otimes Q_{uu} \preceq 0.$$

Thus, because of $Q_{uu} \succ 0$, what remains to show is

$$\hat{\Gamma}^{\mathsf{T}}\hat{\Gamma} \preceq P^{\mathsf{T}}\Gamma^{\mathsf{T}}\Gamma P.$$

This is proven by the fact that

$$x^{\mathsf{T}}(P^{\mathsf{T}}\Gamma^{\mathsf{T}}\Gamma P - \hat{\Gamma}^{\mathsf{T}}\hat{\Gamma})x = \|y\|^2 - \|Py\|^2 \geq 0$$

for all $x \in \mathbb{R}^{\hat{N}}$, where $y := \Gamma P x$. ■

Theorem 1 implies that the regularity of the original networked dissipative system, or equivalently, the existence of the structured Lyapunov function in (10) is key to guaranteeing stability preservation in the clustered model reduction. Note that the stability of approximate models can be robustly guaranteed regardless of the amplitude of approximation errors.

2) Approximation Error Analysis: Next, we analyze approximation errors caused by the clustered model reduction. It has been shown in [5] that the $\mathcal{H}_2$-norm in (16) is bounded as

$$\|\boldsymbol{G}(s) - \hat{\boldsymbol{G}}(s)\|_{\mathcal{H}_2} \leq \|\boldsymbol{\Xi}(s)\|_{\mathcal{H}_\infty} \|\overline{\boldsymbol{P}}^{\mathsf{T}}\boldsymbol{X}(s)\|_{\mathcal{H}_2} \tag{18}$$

where $\boldsymbol{X}$ and $\boldsymbol{\Xi}$ are both stable and defined by

$$\begin{aligned} \boldsymbol{X}(s) &:= (sI - \boldsymbol{A})^{-1}\boldsymbol{B}, \\ \boldsymbol{\Xi}(s) &:= \boldsymbol{C}\boldsymbol{P}(sI - \boldsymbol{P}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{P})^{-1}\boldsymbol{P}^{\mathsf{T}}\boldsymbol{A} + \boldsymbol{C}. \end{aligned} \tag{19}$$

In addition, it can be shown that the $\mathcal{H}_\infty$-norm of $\boldsymbol{\Xi}$ is bounded by a constant determined by $\boldsymbol{A}$ and $\boldsymbol{C}$. In particular, it is bounded by $\gamma > 0$ such that

$$\begin{bmatrix} \boldsymbol{A}^{\mathsf{T}} + \boldsymbol{A} & \frac{1}{\sqrt{\gamma}}\boldsymbol{A} & \boldsymbol{C}^{\mathsf{T}} \\ * & -I & \frac{1}{\gamma}\boldsymbol{C}^{\mathsf{T}} \\ * & * & -I \end{bmatrix} \prec 0. \tag{20}$$

Note that such a constant γ exists whenever $\boldsymbol{\Sigma}$ is regular.

On the other hand, to analyze the $\mathcal{H}_2$-norm of $\overline{\boldsymbol{P}}^{\mathsf{T}}\boldsymbol{X}$, we use the controllability gramian $\boldsymbol{\Phi} \succeq 0$, which is the solution of the Lyapunov equation

$$\boldsymbol{A}\boldsymbol{\Phi} + \boldsymbol{\Phi}\boldsymbol{A}^{\mathsf{T}} + \boldsymbol{B}\boldsymbol{B}^{\mathsf{T}} = 0. \tag{21}$$

In the following arguments, we denote the summation of principal submatrices by

$$\mathcal{D}_n(M) := M_1 + \cdots + M_N \tag{22}$$

where $M_i \in \mathbb{R}^{n\times n}$ denotes the ith principal submatrix of $M \in \mathbb{R}^{Nn\times Nn}$. Then we show the following lemma.

Lemma 2: Given a networked dissipative system $\boldsymbol{\Sigma}$ in (12), let $\boldsymbol{\Phi} \succeq 0$ be the solution of (21). Define

$$\boldsymbol{\Psi} := \boldsymbol{E}^{\mathsf{T}}(I \otimes H)^{\mathsf{T}}\boldsymbol{\Phi}(I \otimes H)\boldsymbol{E} \tag{23}$$

where $\boldsymbol{E} := [I \otimes e_1^{\hat{n}}, \ldots, I \otimes e_{\hat{n}}^{\hat{n}}]$. Then

$$\|\overline{\boldsymbol{P}}^{\mathsf{T}}\boldsymbol{X}(s)\|_{\mathcal{H}_2} = \sqrt{\operatorname{tr}\overline{H}^{\mathsf{T}}\mathcal{D}_n(\boldsymbol{\Phi})\overline{H} + \operatorname{tr}\overline{P}^{\mathsf{T}}\mathcal{D}_N(\boldsymbol{\Psi})\overline{P}}, \quad (24)$$

where $\boldsymbol{X}$ is defined as in (19).

Proof: Because the orthogonal complement of $\boldsymbol{P}$ can be expressed as

$$\overline{\boldsymbol{P}} = [\overline{P} \otimes H, I \otimes \overline{H}], \quad (25)$$

the square of the $\mathcal{H}_2$-norm in (24) is equal to

$$\operatorname{tr}\overline{\boldsymbol{P}}^{\mathsf{T}}\boldsymbol{\Phi}\overline{\boldsymbol{P}} = \operatorname{tr}(I\otimes\overline{H})^{\mathsf{T}}\boldsymbol{\Phi}(I\otimes\overline{H}) + \operatorname{tr}(\overline{P}\otimes H)^{\mathsf{T}}\boldsymbol{\Phi}(\overline{P}\otimes H).$$

For the first term, from the definition in (22), we see that

$$\operatorname{tr}(I \otimes \overline{H})^{\mathsf{T}}\boldsymbol{\Phi}(I \otimes \overline{H}) = \operatorname{tr}\overline{H}^{\mathsf{T}}\mathcal{D}_n(\boldsymbol{\Phi})\overline{H}.$$

On the other hand, because of $\boldsymbol{E}\boldsymbol{E}^{\mathsf{T}} = I$, the second term is expressed as

$$\operatorname{tr}\boldsymbol{E}^{\mathsf{T}}(\overline{P}\,\overline{P}^{\mathsf{T}} \otimes I)\boldsymbol{E}\boldsymbol{E}^{\mathsf{T}}(I \otimes H)^{\mathsf{T}}\boldsymbol{\Phi}(I \otimes H)\boldsymbol{E}. \quad (26)$$

By direct calculation, we can verify that

$$\boldsymbol{E}^{\mathsf{T}}(\overline{P}\,\overline{P}^{\mathsf{T}} \otimes I)\boldsymbol{E} = I \otimes \overline{P}\,\overline{P}^{\mathsf{T}}.$$

Thus, (26) is equal to

$$\operatorname{tr}(I \otimes \overline{P})^{\mathsf{T}}\boldsymbol{\Psi}(I \otimes \overline{P}) = \operatorname{tr}\overline{P}^{\mathsf{T}}\mathcal{D}_N(\boldsymbol{\Psi})\overline{P},$$

which proves the claim. ■

Lemma 2 shows that the $\mathcal{H}_2$-norm of $\overline{\boldsymbol{P}}^{\mathsf{T}}\boldsymbol{X}$ can be analyzed on the basis of $\mathcal{D}_n(\boldsymbol{\Phi})$ and $\mathcal{D}_N(\boldsymbol{\Psi})$. These matrices can be used to find an orthogonal projector H and an aggregation matrix P, as in the following procedure. The optimal projector H minimizing the first term in (24) can be found by the eigendecomposition of $\mathcal{D}_n(\boldsymbol{\Phi})$. In particular, let $\hat{n}$ be the dimension of approximated subsystems, and let $\lambda_i \geq 0$ denote the ith largest eigenvalue of $\mathcal{D}_n(\boldsymbol{\Phi})$. Then, it follows that

$$\min_{H\in\mathcal{P}^{n\times\hat{n}}} \operatorname{tr}\overline{H}^{\mathsf{T}}\mathcal{D}_n(\boldsymbol{\Phi})\overline{H} = \lambda_{\hat{n}+1} + \cdots + \lambda_n. \quad (27)$$

The minimizer H is given as the matrices composed of the eigenvectors associated with $\lambda_1, \ldots, \lambda_{\hat{n}}$, i.e.

$$\mathcal{D}_n(\boldsymbol{\Phi})H = H\operatorname{diag}(\lambda_1, \ldots, \lambda_{\hat{n}}). \quad (28)$$

On the other hand, to analyze the second term in (24), we use the expression of

$$\operatorname{tr}\overline{P}^{\mathsf{T}}\mathcal{D}_N(\boldsymbol{\Psi})\overline{P} = \sum_{l=1}^{\hat{N}}\left\{\operatorname{tr}\boldsymbol{\Psi}_{\mathcal{I}_{[l]}} - p_{[l]}^{\mathsf{T}}\boldsymbol{\Psi}_{\mathcal{I}_{[l]}}p_{[l]}\right\} \quad (29)$$

where $\boldsymbol{\Psi}_{\mathcal{I}_{[l]}}$ is the principal submatrix of $\mathcal{D}_N(\boldsymbol{\Psi})$ compatible with $\mathcal{I}_{[l]}$, i.e.

$$\boldsymbol{\Psi}_{\mathcal{I}_{[l]}} := (e_{\mathcal{I}_{[l]}}^N)^{\mathsf{T}}\mathcal{D}_N(\boldsymbol{\Psi})e_{\mathcal{I}_{[l]}}^N. \quad (30)$$

This expression leads to the following notion of cluster reducibility.

Definition 3: In the same notation as that in Lemma 2, a cluster $\mathcal{I}_{[l]}$ is said to be θ_l-*reducible* if

$$\operatorname{tr}\boldsymbol{\Psi}_{\mathcal{I}_{[l]}} - p_{[l]}^{\mathsf{T}}\boldsymbol{\Psi}_{\mathcal{I}_{[l]}}p_{[l]} \leq \theta_l \quad (31)$$

where $\boldsymbol{\Psi}_{\mathcal{I}_{[l]}}$ is defined as in (30).

On the basis of Definition 3, we construct a cluster set $\{\mathcal{I}_{[l]}\}_{l\in\mathbb{L}}$ such that each of clusters is θ_l-reducible; see Section III-C for a sequential clustering algorithm. For such a cluster set, it readily follows that (29) is bounded by the sum of θ_l.

As summarizing all the facts shown above, we state the following theorem for the clustered model reduction of networked dissipative systems.

Theorem 2: Given a regular networked dissipative system $\boldsymbol{\Sigma}$ in (12), let $\gamma > 0$ be such that (20) holds. In the same notation as that in Lemma 2, if the orthogonal projector H satisfies (28) and the aggregation matrix P is given such that each of clusters $\mathcal{I}_{[l]}$ is θ_l-reducible, then the approximate model $\hat{\boldsymbol{\Sigma}}$ in (13) is a regular networked dissipative system and it satisfies

$$\|\boldsymbol{G}(s) - \hat{\boldsymbol{G}}(s)\|_{\mathcal{H}_2} \leq \gamma\sqrt{\textstyle\sum_{i=\hat{n}+1}^{n}\lambda_i + \sum_{l=1}^{\hat{N}}\theta_l} \quad (32)$$

where $\boldsymbol{G}$ and $\hat{\boldsymbol{G}}$ are defined as in (17).

C. Remarks on Implementation

In this subsection, we give some remarks on the implementation of the proposed clustered model reduction. As shown in Lemma 2, our method is based on using $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ as indices to construct a structured projector $\boldsymbol{P}$. Note that the orthogonal projector H is involved into $\boldsymbol{\Psi}$ in (23), which is used to find an aggregation matrix P. This implies that we need to determine H before finding P. Such an order of implementation comes from the fact that the approximation error is analyzed on the premise of $\overline{\boldsymbol{P}}$ in (25). On the other hand, we notice that it admits the alternative expression of

$$\overline{\boldsymbol{P}} = [\overline{P} \otimes I, P \otimes \overline{H}].$$

From this, we can show that the same equation in (24) holds when we replace $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ with

$$(P \otimes I)^{\mathsf{T}}\boldsymbol{\Phi}(P \otimes I), \quad \tilde{\boldsymbol{E}}^{\mathsf{T}}\boldsymbol{\Phi}\tilde{\boldsymbol{E}},$$

respectively, where $\tilde{\boldsymbol{E}} := [I \otimes e_1^n, \ldots, I \otimes e_n^n]$. These indices are suitable for determining P before H.

Next, we consider constructing a set of reducible clusters. From an argument similar to (27), we notice for (31) that

$$\min_{p_{[l]}}\{\operatorname{tr}\boldsymbol{\Psi}_{\mathcal{I}_{[l]}} - p_{[l]}^{\mathsf{T}}\boldsymbol{\Psi}_{\mathcal{I}_{[l]}}p_{[l]}\} = \operatorname{tr}\boldsymbol{\Psi}_{\mathcal{I}_{[l]}} - \lambda_{\max}(\boldsymbol{\Psi}_{\mathcal{I}_{[l]}}), \quad (33)$$

which is expected to be small when the rank of $\boldsymbol{\Psi}_{\mathcal{I}_{[l]}}$ is close to one. On the basis of (33), we propose the following clustering algorithm, where, by letting the initial clusters as $\mathcal{I}_{[i]} = i$ for $i \in \{1, \ldots, N\}$, we sequentially decrease the number of clusters by merging two of them iteratively. Let an index matrix $\mathcal{D}_N(\boldsymbol{\Psi})$ be given and consider the situation where a temporary cluster set has been formed. For the

sequential clustering, we find a pair of clusters $\mathcal{I}_{[i]}$ and $\mathcal{I}_{[j]}$ such that they attain the minimum of

$$\operatorname{tr} \boldsymbol{\Psi}_{\mathcal{I}_{[i]}\cup\mathcal{I}_{[j]}} - \lambda_{\max}(\boldsymbol{\Psi}_{\mathcal{I}_{[i]}\cup\mathcal{I}_{[j]}}),$$

and then we merge the pair of minimizers, i.e. $\mathcal{I}_{[i]}$ is updated to $\mathcal{I}_{[i]} \cup \mathcal{I}_{[j]}$. This is repeated until when a stopping criterion is attained. Finally, for the minimum in (33), denoted by θ_l, the sequential algorithm produces a cluster set such that each of clusters $\mathcal{I}_{[l]}$ is θ_l-reducible. Note that the corresponding aggregation weight $p_{[l]}$ is found as the eigenvector associated with $\lambda_{\max}(\boldsymbol{\Psi}_{\mathcal{I}_{[l]}})$, which is the minimizer of (33).

IV. Numerical Experiments

A. System Description

In this section, we demonstrate the efficiency of the proposed clustered model reduction method. Let us consider giving the subsystem dynamics as an interconnected second order system of

$$\begin{cases} M\ddot{\xi}_i + D\dot{\xi}_i + K\xi_i = Fu_i \\ y_i = F^{\mathsf{T}}\dot{\xi}_i \end{cases} \tag{34}$$

where $M \succ 0$ denotes a diagonal mass matrix, $D \succ 0$ denotes a diagonal damper matrix, $K \succ 0$ denotes a spring stiffness matrix, and F is a matrix having a compatible dimension. For $x_i := [\xi_i^{\mathsf{T}}, \dot{\xi}_i^{\mathsf{T}}]^{\mathsf{T}}$, the subsystem dynamics can be represented as Σ_i in (1) with

$$\begin{aligned} A &= \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix}, \\ C &= \begin{bmatrix} 0 & F^{\mathsf{T}} \end{bmatrix}, \end{aligned} \tag{35}$$

whose dimension is denoted by n. Note that this Σ_i is passive since the input and output are collocated [20]. Furthermore, we give the interconnection matrix Γ in (4) as the negative of the unweighted graph Laplacian of the Holme-Kim model [1], which is known as an extension of the Barabasi-Albert model describing complex networks. As shown in Section II, this class of interconnected passive systems is stable and can be regularized without destroying their interconnection structure. Finally, the input and output matrices in (12) are given as

$$\boldsymbol{B} = [e_1^N, e_2^N, e_3^N] \otimes B, \quad \boldsymbol{C} = I,$$

which imply that a set of external input signals is applied to three of N subsystems and the evaluation output is taken as the states of all subsystems.

B. Results of Clustered Model Reduction

We first show the result when giving the number of subsystems as $N = 50$ and the dimension of each subsystem as $n = 10$, which yield a 500-dimensional interconnected system. The interconnection structure of Γ is depicted in Fig. 1, where the subsystems are denoted by the nodes and the external inputs are applied to the arrowed ones. Furthermore, instead of specifying the system parameter matrices in (35), we plot the Bode diagram of Σ_i by the solid lines in Fig. 2. Using our clustered model reduction for this system, we obtain the result in Fig. 3, where the relative

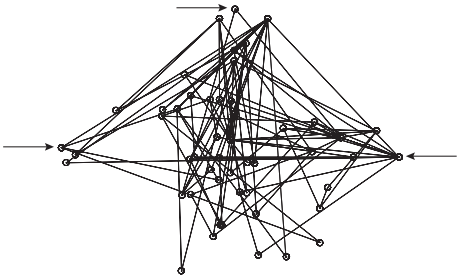

Fig. 1. Original interconnection structure.

approximation errors, defined as $\|\boldsymbol{G} - \hat{\boldsymbol{G}}\|_{\mathcal{H}_2}/\|\boldsymbol{G}\|_{\mathcal{H}_2}$, are plotted versus the dimensions of approximate models. In particular, the line with diamonds corresponds to the case where we implement the sequential clustering algorithm in Section III-C, while the approximation by the orthogonal projection is not applied, i.e. $H = I$ in (14). The lines with circles and squares correspond to the cases where the sequential clustering is performed after applying orthogonal projection by $H \in \mathbb{R}^{10\times 8}$ and $H \in \mathbb{R}^{10\times 6}$ that reduces the dimensions of subsystems to $\hat{n} = 8$ and $\hat{n} = 6$, respectively. From this figure, we see that the case of $H = I$ provides the best approximation precision among them.

Next, we show the result for $N = 10$ and $n = 50$, which yield another 500-dimensional interconnected system. The interconnection structure of Γ is depicted in Fig. 4 and the Bode diagram of Σ_i is shown by the dotted lines in Fig. 2. In this case, since the redundancy of the subsystem dynamics may be larger than the previous setting, orthogonal projection for subsystem approximation is expected to be more effective. The resultant approximation errors are plotted in Fig. 5, where we use the notation similar to that in Fig. 3. In particular, the line with diamonds corresponds to the case where we only implement the sequential clustering, while the lines with circles and squares correspond to the cases where the sequential clustering is performed after applying orthogonal projection by $H \in \mathbb{R}^{50\times 40}$ and $H \in \mathbb{R}^{50\times 30}$ that reduces the dimensions of subsystems to $\hat{n} = 40$ and $\hat{n} = 30$. From this figure, we see that the three lines cross each other. This means that there is an appropriate choice of reduced dimensions determined by clustering and orthogonal projection depending on the resultant dimension of approximate models.

V. Concluding Remarks

In this paper, we proposed a clustered model reduction method for networked dissipative systems, where both network structure simplification and subsystem approximation are performed. To develop such a clustered model reduction method, we derived a condition for the existence of block-diagonal Lyapunov functions for networked dissipative systems, which is essential for stability preservation in the clustered model reduction. In addition, we performed an approximation error analysis in terms of the $\mathcal{H}_2$-norm.

In the proposed clustered model reduction, orthogonal projection is used for subsystem approximation. On the basis

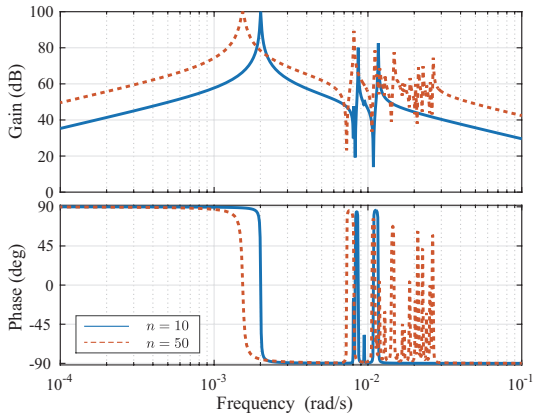

Fig. 2. Bode diagrams of subsystems.

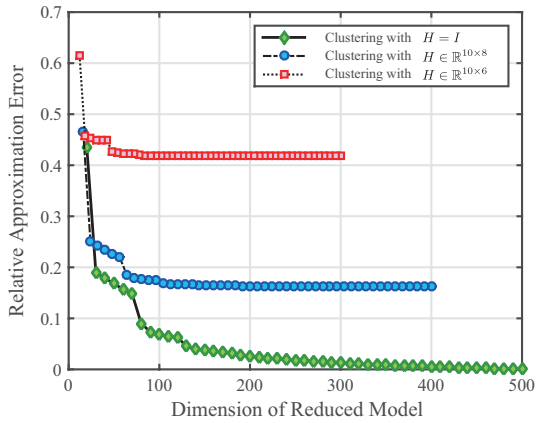

Fig. 3. Approximation errors versus dimensions of approximate models.

of the fact that singular perturbation approximation admits a projection-like formula shown in [18], [19], it would be possible to generalize the result for orthogonal projection to that for singular perturbation approximation. This is one of future works to pursue.

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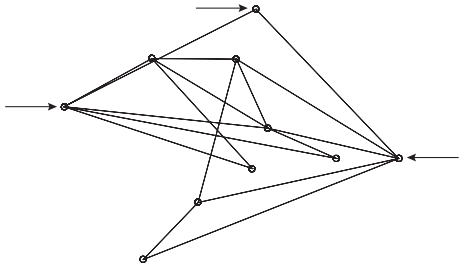

Fig. 4. Original interconnection structure.

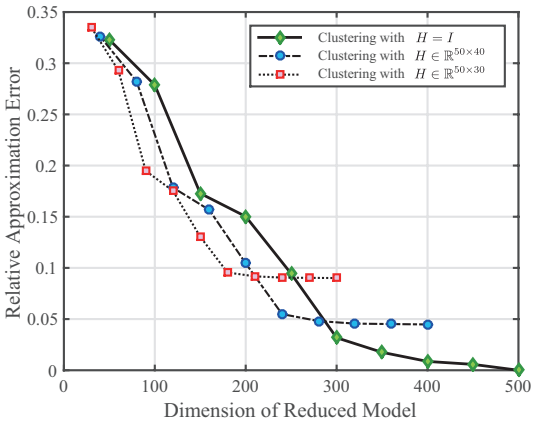

Fig. 5. Approximation errors versus dimensions of approximate models.

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