Convex Gradient Controller Design for Incrementally Passive Systems with Quadratic Storage Functions

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Abstract— **In this paper, we give an equivalence condition for incremental passivity in terms of convex gradients and perform output regulator design for incrementally passive systems. To derive the equivalence condition, we focus on the class of incremental passive systems with quadratic storage functions, which can be transformed into a particular realization called a self-dual realization. On the basis of the self-dual realization, in which the input matrix necessarily coincides with the transpose of the output matrix, we show that the convexity of potential functions is necessary and sufficient for the incremental passivity of systems whose vector field is given as the gradient of potential functions. Furthermore, we show that the equilibrium of such convex gradient systems can be analyzed via the convex conjugate defined by the Legendre-Fenchel transformation. Combining these facts, we then develop a design method of output regulators that have a potential to improve a degree of stability while leaving the original equilibrium of integrator-based control systems invariant. The stability improvement is demonstrated though an example of power systems control, in which the resultant output regulator is shown to have a low-pass property with the nonlinearity of input saturation.**

I. Introduction

In the real world, there can be found a number of systems whose behavior is described as the gradient of physical (or perhaps virtual) energy functions. Examples include reaction-diffusion systems, Euler-Lagrange systems, port-Hamiltonian systems, and so forth [1], [2]. As another example of such gradient systems, dynamics stemming from the gradient of objective functions can be found in the optimization theory [3]. In particular, a relation between optimization-based gradient systems and power system control has also been investigated recently [4]. Generally speaking, analyses on fundamental system properties, such as stability and existence of equilibria, would be a necessary first step to aim at systematically controlling them.

As for stability analyses, it is well known that passivity is one of tractable properties of physical systems to discuss the stability of interconnected systems. More specifically, negative feedback interconnection of passive components retains the passivity, thereby proving the stability of feedback control systems. By making use of this fact, various types of stability analyses based on passivity have been performed in the literature [5], [6]. For example, analysis and synthesis methods for network synchronization and coordination have been developed in [7], [8] as an application of such passivity-based stability analyses.

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For linear systems, the passivity can be characterized by linear matrix inequalities, which are useful for both theoretical and numerical analyses. On the other hand, for general nonlinear systems, it is however difficult to find out a simple characterization of passivity, except for the port-Hamiltonian systems known as a useful class of passive nonlinear systems. In view of this, it is valuable to explore a simple characterization towards systematic control system design.

Against this background, this paper aims at giving a tractable characterization of passive systems. In particular, focusing on the class of incrementally passive systems with quadratic storage functions, which can be transformed into a particular realization called a self-dual realization [9], we show that the convexity of potential functions is necessary and sufficient for the incremental passivity of systems whose vector field is given as the gradient of potential functions. This clarification is based on the fact that the input matrix necessarily coincides with the transpose of the output matrix in the self-dual realization. Furthermore, we show that the equilibrium of such convex gradient systems can be systematically analyzed via the conjugate of convex functions, defined by the Legendre-Fenchel transformation [10]. Note that the incremental passivity [11]–[13], which is generally stronger than the standard notion of passivity, is a useful property to guarantee the stability of an equilibrium of interconnected nonlinear systems, especially in the case where the equilibrium is not at the origin of the state-space.

On the basis of the investigations above, we then develop a design method of output regulators that have a potential to improve a stability degree while leaving the original equilibrium of integrator-based control systems invariant. The efficiency of the proposed method is numerically demonstrated through an example of decentralized output regulation of swing equations appearing in power systems, where the resultant output regulator is shown to have a low-pass property with the nonlinearity of input saturation. It should be noted that input saturation generally causes the loss of passivity as shown in [14]. In contrast to this, we show that, by giving an appropriate convex function, a low-pass controller subject to input saturation can be expressed as a convex gradient system possessing incremental passivity. Furthermore, we demonstrate that a degree of system stability can be improved as long as the time constant of controllers is sufficiently small, i.e., as long as a singular perturbation approximation of controllers is valid, even when the control input is subject to saturation.

Finally, we give some references related to passivity-based analyses of dynamical systems originating in convex optimization. A passive controller design method for network flow optimization is proposed in [15], where a dynamics stemming from a source updating law is shown to have a passive property. It should be noted that the paper only deals with a specific problem where the objective function is given as the sum of functions of scalar variable, which leads to a network of one-dimensional subsystems. Therefore, the result cannot be applied straightforwardly to general optimization problems that involve multivariate objective functions, yielding a network of multi-dimensional subsystems. In a similar way, [16], [17] propose a passive controller design method for market mechanisms in smart grids. Furthermore, [18] shows that a class of passivity-based cooperative control problems has a relation with network flow optimization problems. In that paper, the convergence of output agreement is discussed on the premise that subsystems forming a cooperative network are assumed to be passive.

The remainder of this paper is structured as follows. In Section II, summarizing the preliminary facts for incremental passivity and convex functions, we give an equivalence condition of incrementally passive systems in terms of convex gradients. Furthermore, we show that the equilibrium of convex gradient systems can be analyzed by conjugating convex functions. Next, in Section III, we apply the results in Section II to output regulator design for incrementally passive systems, where we show that a monotone property of convex functions can be utilized to improve a stabilizing performance of control systems. In Section IV, numerical simulations are given to show the efficiency of the proposed output regulator design method. Finally, concluding remarks are provided in Section V.

II. Analysis of Convex Gradient Systems

A. Preliminaries

1) Incrementally Passive Systems: Let us consider the class of dynamical systems given by

$$\Sigma : \begin{cases} \dot{x} = f(x) + Bu \\ y = Cx \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state, $u \in \mathbb{R}^p$ denotes the input signal, and $y \in \mathbb{R}^p$ denotes the output signal. In the following, we denote the domains of x, u, and y by $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$, respectively. Furthermore, let $\mathcal{E}_\Sigma$ denote the set of all triplets $(x, u, y) \in \mathcal{X} \times \mathcal{U} \times \mathcal{Y}$ that satisfy the dynamical equation of Σ in (1). In this notation, we first introduce the following notion of incremental passivity [11]–[13].

Definition 1: A dynamical system Σ in (1) is said to be *incrementally passive* if there exist a positive definite function $S : \mathbb{R}^n \to \mathbb{R}$ and a positive semidefinite function $\delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\dot{S}(\Delta x) \leq \Delta u^{\mathsf{T}} \Delta y - \delta(x_1, x_2), \quad t \geq 0 \tag{2}$$

for any $(x_1, u_1, y_1) \in \mathcal{E}_\Sigma$ and $(x_2, u_2, y_2) \in \mathcal{E}_\Sigma$, where

$$\Delta x := x_1 - x_2, \quad \Delta u := u_1 - u_2, \quad \Delta y := y_1 - y_2.$$

In particular, it is said to be *strictly incrementally passive* if (2) is satisfied with δ being positive unless $x_1 = x_2$.

One major advantage of the incremental passivity is that a network system composed of the negative feedback interconnection of incrementally passive systems is shown to have a stable nonzero equilibrium. This can be seen as follows. Let Σ_1 and Σ_2 be incrementally passive systems with respect to the storage functions $S_1 : \mathbb{R}^{n_1} \to \mathbb{R}$ and $S_2 : \mathbb{R}^{n_2} \to \mathbb{R}$, and suppose that at least one of them is strictly incrementally passive. Furthermore, let $(x_1, u_1, y_1) \in \mathcal{E}_{\Sigma_1}$ and $(x_2, u_2, y_2) \in \mathcal{E}_{\Sigma_2}$ be their feasible dynamical trajectories whereas let $(x_1^*, u_1^*, y_1^*) \in \mathcal{E}_{\Sigma_1}$ and $(x_2^*, u_2^*, y_2^*) \in \mathcal{E}_{\Sigma_2}$ be their *stationary* trajectories. In this notation, if Σ_1 and Σ_2 are interconnected with

$$u_1 = y_2, \quad u_2 = -y_1,$$

then for $S(x_1, x_2) := S_1(x_1) + S_2(x_2)$, which is positive definite, we have

$$\begin{aligned}\dot{S}(x_1 - x_1^*, x_2 - x_2^*) &= \dot{S}_1(x_1 - x_1^*) + \dot{S}_2(x_2 - x_2^*) \\ &\leq (u_1 - u_1^*)^{\mathsf{T}}(y_1 - y_1^*) + (u_2 - u_2^*)^{\mathsf{T}}(y_2 - y_2^*) = 0,\end{aligned}$$

where the strict inequality holds unless $x_1 = x_1^*$ and $x_2 = x_2^*$. This implies that S serves as a Lyapunov function to prove the asymptotic stability of the equilibrium (x_1^*, x_2^*).

2) Convex Functions: Next, we give some preliminary facts for convex functions [3], [10]. A function $F : \mathbb{R}^n \to \mathbb{R}$ is said to be *convex* if

$$F\left((1-\lambda)x + \lambda x'\right) \leq (1-\lambda)F(x) + \lambda F(x') \tag{3}$$

for all $0 < \lambda < 1$ and for all pairs of x and x' in the domain such that F is finite. Such a domain, denoted by $\mathcal{X}_F$, is called the *effective domain* of F and it is known to be convex. In particular, F is said to be *strictly convex* if (3) holds with the strict inequality unless $x = x'$.

For F being continuously differentiable, the *gradient* of F is denoted by ∇F, whose ith element is given by $\frac{\partial F}{\partial x_i}$ with x_i denoting the ith element of x. By the gradient of F, the convexity can be characterized by

$$F(x) - F(x') \geq \nabla F^{\mathsf{T}}(x')(x - x') \tag{4}$$

for all $x, x' \in \mathcal{X}_F$. Furthermore, for F being twice continuously differentiable, the *Hessian* of F is denoted by H_F, whose (i,j)-element is given by $\frac{\partial^2 F}{\partial x_i \partial x_j}$. The convexity of F can also be characterized by the positive semidefiniteness of the Hessian as

$$H_F(x) \succeq 0 \tag{5}$$

for all $x \in \mathcal{X}_F$. In a similar manner, the strict notion of convexity can be characterized by the strict inequalities of (4) and (5). In the rest of this paper, for simplicity, all convex functions under consideration are assumed to be twice continuously differentiable.

In many applications of convex analysis theory, such as convex optimization, the following conjugate transformation is often utilized for dual analyses. The *conjugate* of F is defined by

$$\overline{F}(z) := \sup_{x\in\mathcal{X}} \left\{ z^\mathsf{T} x - F(x) \right\}, \tag{6}$$

where $\mathcal{X}$ is a convex domain. It is known that, as long as F is convex, $\overline{F}$ is also convex and the conjugate of $\overline{F}$ coincides with F, namely

$$F(x) = \sup_{z\in\mathcal{Z}} \left\{ x^\mathsf{T} z - \overline{F}(z) \right\}, \tag{7}$$

where $\mathcal{Z}$ is the set of all z such that the supremum of (7) is finite. Furthermore, the strict convexity of F is equivalent to the smoothness of $\overline{F}$. In convex analysis theory, the transformation between (6) and (7) is called the Legendre-Fenchel transformation, and some of conjugate pairs can be found in closed forms.

B. An Equivalence Condition for Incremental Passivity

In this subsection, we derive an equivalence condition for incremental passivity. This derivation is based on a specific realization having a symmetric nature, called a *self-dual realization* [9] for the case of linear systems, as shown in the following lemma.

Lemma 1: Let Σ in (1) be given and assume that it is incrementally passive (strictly incremental passive) with respect to $S(x) = \frac{1}{2}x^\mathsf{T} V x$, where V is positive definite. Furthermore, let V_c be a Cholesky factor of V such that $V = V_\mathrm{c}^\mathsf{T} V_\mathrm{c}$. Then, the realization of

$$\tilde{\Sigma} : \begin{cases} \dot{\tilde{x}} = V_\mathrm{c} f(V_\mathrm{c}^{-1}\tilde{x}) + V_\mathrm{c} B u \\ y = C V_\mathrm{c}^{-1} \tilde{x} \end{cases} \tag{8}$$

is incrementally passive (strictly incremental passive) with respect to $\tilde{S}(\tilde{x}) = \frac{1}{2}\|\tilde{x}\|^2$. In addition, $V_\mathrm{c} B = (C V_\mathrm{c}^{-1})^\mathsf{T}$.

Proof: To prove the claim for incremental passivity, we can assume that $\delta = 0$ without loss of generality. Then, for $(x_1, u_1, y_1) \in \mathcal{E}_\Sigma$ and $(x_2, u_2, y_2) \in \mathcal{E}_\Sigma$, the incremental inequality in (2) is represented by

$$\left\{ f(x_1) - f(x_2) \right\}^\mathsf{T} V \Delta x + (B\Delta u)^\mathsf{T} V \Delta x \le \Delta u^\mathsf{T} \Delta y.$$

By the coordinate transformation of $\tilde{x} = V_\mathrm{c} x$, we have

$$\left\{ \tilde{f}(\tilde{x}_1) - \tilde{f}(\tilde{x}_2) \right\}^\mathsf{T} \Delta\tilde{x} + (\tilde{B}\Delta u)^\mathsf{T} \Delta\tilde{x} \le \Delta u^\mathsf{T} \Delta y \tag{9}$$

where $\Delta\tilde{x} := V_\mathrm{c}\Delta x$, $\tilde{f}(\tilde{x}) := V_\mathrm{c} f(V_\mathrm{c}^{-1}\tilde{x})$, and $\tilde{B} := V_\mathrm{c} B$. This implies that $\tilde{\Sigma}$ in (8) is incrementally passive with respect to $\tilde{S}(\tilde{x}) = \frac{1}{2}\|\tilde{x}\|^2$. Note that (9) is satisfied for all feasible trajectories of $x_1, x_2 \in \mathcal{X}$ and $u_1, u_2 \in \mathcal{U}$. Thus, it follows for $\tilde{C} := C V_\mathrm{c}^{-1}$ that

$$\left\{ \tilde{f}(\tilde{x}_1) - \tilde{f}(\tilde{x}_2) \right\}^\mathsf{T} \Delta\tilde{x} \le 0, \quad \Delta u^\mathsf{T} (\tilde{B}^\mathsf{T} - \tilde{C}) \Delta\tilde{x} \le 0.$$

This leads to $\tilde{B} = \tilde{C}^\mathsf{T}$, because $\Delta\tilde{x}$ and Δu are arbitrary. The same argument with the change of variables of δ proves the claim for the strict notion of incremental passivity. ∎

Lemma 1 shows that any incrementally passive systems with a quadratic storage function can be transformed into the specific realization given as in (8). For linear systems, i.e., $f(x) = Ax$, the specific realization satisfies that

$$A + A^\mathsf{T} \preceq 0, \quad B = C^\mathsf{T}.$$

In the literature, this particular realization of passive systems is known to have good compatibility with a passive controller synthesis, model reduction, and so forth [9], [19]. The facts of $B = C^\mathsf{T}$ and the storage function of $S(x) = \frac{1}{2}\|x\|^2$ in the self-dual realization are essential in the following arguments.

Next, on the basis of the self-dual realization, we derive an equivalence condition for incremental passivity in the context of convex gradient. To this end, the following fact for convex functions is useful [3].

Lemma 2: Let $F : \mathbb{R}^n \to \mathbb{R}$ and define

$$\delta_F(x_1, x_2) := \left\{ \nabla F(x_1) - \nabla F(x_2) \right\}^\mathsf{T} (x_1 - x_2). \tag{10}$$

Then, δ_F is nonnegative for all $x_1, x_2 \in \mathcal{X}_F$ if and only if F is convex. In particular, δ_F is positive unless $x_1 = x_2$ if F is strictly convex.

Lemma 2 shows that the positive semidefiniteness of δ_F in (10), appearing in the incremental passivity analysis, is equivalent to the convexity of F. This equivalence leads to the following result.

Theorem 1: Let Σ in (1) be given and assume that there exist a function $F : \mathbb{R}^n \to \mathbb{R}$ and a skew symmetric matrix $J \in \mathbb{R}^{n\times n}$ such that

$$f(x) = -\nabla F(x) + Jx. \tag{11}$$

Then, Σ is incrementally passive with respect to $S(x) = \frac{1}{2}\|x\|^2$ if and only if F is convex. In particular, Σ is strictly incrementally passive if F is strictly convex.

Proof: From Lemma 1, we see that $B = C^\mathsf{T}$ holds if Σ is incrementally passive with respect to $S(x) = \frac{1}{2}\|x\|^2$. In the notation of (2), we have

$$\dot{S}(\Delta x) - \Delta u^\mathsf{T} \Delta y = -\delta_F(x_1, x_2),$$

where δ_F is defined as in (10). Thus, by Lemma 2, it is nonpositive for all $x_1, x_2 \in \mathcal{X}_F$ if and only if F is convex. Similarly, the claim for strictly incremental passivity is proven by the strict convexity of F, which implies that δ_F is positive unless $x_1 = x_2$. ∎

Theorem 1 shows that, as long as the vector field is represented by the gradient of potential functions, the convexity of potential functions is necessary and sufficient for incremental passivity with respect to quadratic storage functions. On the basis of this investigation, let us introduce the following terminology.

Definition 2: A dynamical system Σ in (1) is said to be a *convex gradient system* if there exist a convex function $F : \mathbb{R}^n \to \mathbb{R}$ and a skew symmetric matrix $J \in \mathbb{R}^{n\times n}$ such that (11) holds. In particular, a convex gradient system Σ is said to be *self-dual* if $B = C^\mathsf{T}$.

From the discussion above, it can be seen that a self-dual convex gradient system is incrementally passive with respect to the unitary quadratic storage function. It should be noted that any linear passive system is equivalent to a convex gradient system, because its self-dual realization is given by letting

$$F(x) = -\frac{1}{2}x^\mathsf{T}(A + A^\mathsf{T})x, \quad J = A - A^\mathsf{T},$$

which yields $f(x) = Ax$. From this viewpoint, we see that the class of convex gradient systems in Definition 2 involves that of linear passive systems associated with quadratic storage functions.

An interconnected system given by the negative feedback of self-dual convex gradient systems again belongs to the class of convex gradient systems. This can be seen as

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} -\nabla F(x) \\ -\nabla G(\xi) \end{bmatrix} + \begin{bmatrix} J & C^\mathsf{T} H \\ -H^\mathsf{T} C & J' \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix}, \qquad (12)$$

where F and G are convex, J and J' are skew symmetric, and C and H are matrices having compatible dimensions. This type of negative feedback systems can appear in solving convex optimization by gradient methods. To see this more clearly, let us consider the convex-concave optimization

$$\sup_{\xi \in \mathcal{X}_G} \inf_{x \in \mathcal{X}_F} \left\{ F(x) - (H\xi)^\mathsf{T} Cx - G(\xi) \right\}, \qquad (13)$$

which can be viewed as a general form of saddle-point problems arising in the Lagrange dual decomposition for convex optimization [3], [10]. Denoting the objective function by $L(x, \xi)$, we consider the dynamics stemming from the gradient descent and gradient ascent of (13) given as

$$\dot{x} = -\frac{\partial L}{\partial x}(x, \xi), \quad \dot{\xi} = \frac{\partial L}{\partial \xi}(x, \xi), \qquad (14)$$

which corresponds to the Uzawa algorithm to solve convex optimizations. It can be readily verified that (14) is identical to (12) with $J = 0$ and $J' = 0$. Furthermore, the solutions x^* and ξ^* of the convex-concave optimization in (13) coincide with the stable equilibria of (12).

C. Equilibrium Analysis of Convex Gradient Systems

In this subsection, we give a method to analyze the equilibrium of convex gradient systems using the conjugate of convex functions. To this end, we show the following simple but useful fact; see, e.g., [10] for a proof.

Lemma 3: Let a convex gradient system Σ be given as in Definition 2, and assume that F is strictly convex and $J = 0$. Let (x^*, u^*) be a feasible pair such that

$$\nabla F(x^*) = Bu^*. \qquad (15)$$

Then, it follows that

$$x^* = \nabla \overline{F}(Bu^*) \qquad (16)$$

where $\overline{F}$ is the convex conjugate of F defined as in (6).

Lemma 3 shows that the conjugacy of convex functions has a relation to the equilibrium analysis of convex gradient systems. This is based on the fact that ∇F is the inverse function of $\nabla \overline{F}$, and vice versa. For linear systems, it can be verified that (16) is reduced to $x^* = -A^{-1}Bu^*$.

One application of this fact can be found in analyzing a singular perturbation approximation of convex gradient systems. To see this, let us consider applying the singular perturbation approximation to the dynamics of ξ in (12) with $J' = 0$. More specifically, replacing $\dot{\xi}$ with 0, we have the trajectory of ξ along x as

$$\xi = \nabla \overline{G}(-H^\mathsf{T} Cx).$$

Substituting this into the dynamical equation of x in (12), we obtain the singular perturbation model as a convex gradient system

$$\hat{\Sigma} : \dot{x} = -\nabla \hat{F}(x) + Jx \qquad (17)$$

where

$$\hat{F}(x) := F(x) + \overline{G}(-H^\mathsf{T} Cx). \qquad (18)$$

Note that the stable equilibrium of $\hat{\Sigma}$ is identical to that of x in (12). In addition, one can say that a convergence rate to the equilibrium is improved in view of the following stability index.

Definition 3: Let a convex gradient system Σ be given as in Definition 2, and assume that F is strictly convex. Let x^* be a feasible equilibrium such that there exists some u^* satisfying (16). Then, an *incremental stability degree* of Σ with respect to x^* is defined by

$$\theta_\Sigma(x) := \delta_F(x, x^*), \qquad (19)$$

where δ_F is defined as in (10).

The incremental stability degree θ_Σ corresponds to the decreasing rate, i.e., time derivative, of the Lyapunov function $S(x) = \frac{1}{2}\|x - x^*\|^2$, proving the stability of an equilibrium x^*, when injecting the corresponding constant input $u = u^*$ such that (16). For linear systems with A being negative definite, this function measures the degree of negative definiteness because

$$\theta_\Sigma(x) = -(x - x^*)^\mathsf{T} A(x - x^*).$$

On the basis of this definition, we can see for $\hat{\Sigma}$ in (17) that the incremental stability degree with respect to the equilibrium x^* of $\hat{\Sigma}$ becomes larger as

$$\theta_\Sigma(x) \le \theta_{\hat{\Sigma}}(x) \qquad (20)$$

for all $x \in \mathcal{X}_F \cap \mathcal{X}_{\hat{F}}$. The inequality in (20) is proven by the monotone property of convex functions such that any pair of convex functions F_1 and F_2 satisfies

$$\delta_{F_1}(x_1, x_2) \le \delta_{F_1 + F_2}(x_1, x_2) \qquad (21)$$

for all $x_1, x_2 \in \mathcal{X}_{F_2} \cap \mathcal{X}_{F_2}$. This result will be applied to convex gradient regulator design in Section III.

III. Synthesis of Convex Gradient Systems

In this section, we consider designing an output regulator for incrementally passive systems, on the basis of the analyses in Section II. In this output regulator design, we explicitly use the equilibrium analysis based on Lemma 3 to leave the original equilibrium invariant, while guaranteeing the closed-loop stability in a way similar to [13].

Let a convex gradient system be given as in Definition 2 and assume that it is self-dual. Our objective here is to steer the output y towards a constant reference signal y^*. One simple approach to do this is implementing the conventional integrator as

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -\nabla F(x) \\ y^* \end{bmatrix} + \begin{bmatrix} J & C^{\mathsf{T}} \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}. \tag{22}$$

In fact, owing to the incremental passivity of the integrator, the equilibrium (x^*, λ^*) of (22) is proven to be asymptotically stable and $y^* = Cx^*$ is to be guaranteed, as long as F is strictly convex.

However, the convergence rate of (22) may not be desirable because it relies only on the inherent system stability. Towards improving it, one can consider implementing an additional controller to the feedback system. As taking advantage of incremental passivity, it would be reasonable to suppose that the additional input signal u is injected as

$$\Sigma : \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -\nabla F(x) \\ y^* \end{bmatrix} + \begin{bmatrix} J & C^{\mathsf{T}} \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} C^{\mathsf{T}} \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}. \end{cases} \tag{23}$$

To this system, we implement a convex gradient controller with a time constant $\tau > 0$ given as

$$\kappa : \begin{cases} \tau\dot{\xi} = -\nabla G(\xi) + H^{\mathsf{T}} v \\ w = H\xi, \end{cases} \tag{24}$$

which is interconnected with Σ in (23) by

$$v = -y, \quad u = w.$$

Note that it can be transformed to a self-dual realization by scaling the state variable as $\tilde{\xi} = \sqrt{\tau}\xi$.

In this formulation, as the integration of analyses in Section II, we show the following result for designing output regulating systems.

Theorem 2: Consider the feedback system composed of a convex gradient system Σ in (23) and a convex gradient controller κ in (24), and let (x^*, λ^*) denote the equilibrium of (22). If $G : \mathbb{R}^\nu \to \mathbb{R}$ is given as

$$G(\xi) = \mathit{\Gamma}(\xi) - (H^{\mathsf{T}} y^*)^{\mathsf{T}} \xi \tag{25}$$

where $\mathit{\Gamma} : \mathbb{R}^\nu \to \mathbb{R}$ is a strictly convex function such that $\nabla\mathit{\Gamma}(0) = 0$, then

$$\lim_{t\to\infty} (x, \lambda, \xi) = (x^*, \lambda^*, 0) \tag{26}$$

for any initial values of (x, λ, ξ) and any time constant $\tau > 0$. In particular, in the limit of $\tau \to 0$, the resultant feedback system approaches to a convex gradient system $\hat{\Sigma}$ that is given as replacing F in (22) with $\hat{F}$ in (18). Furthermore, for the incremental stability degree with respect to (x^*, λ^*), it follows that

$$\theta_\Sigma(x, \lambda) \le \theta_{\hat{\Sigma}}(x, \lambda) \tag{27}$$

for all $x \in \mathcal{X}_F \cap \mathcal{X}_{\hat{F}}$ and $\lambda \in \mathbb{R}$.

Proof: First, as shown in Lemma 3, we see that $\nabla\overline{\mathit{\Gamma}}$ is the inverse function of $\nabla\mathit{\Gamma}$ and vice versa. Thus, $\nabla\mathit{\Gamma}(0) = 0$ is equivalent to $\nabla\overline{\mathit{\Gamma}}(0) = 0$. Furthermore, from the relation among the affine transformation of conjugates [10], we see that

$$\overline{G}(\mu) = \overline{\mathit{\Gamma}}(\mu + H^{\mathsf{T}} y^*)$$

where $\overline{G}$ and $\overline{\mathit{\Gamma}}$ are the convex conjugates of G and $\mathit{\Gamma}$, respectively. Then, using Theorem 3, we obtain

$$\xi^* = \nabla\overline{G}(-H^{\mathsf{T}} y^*) = \nabla\overline{\mathit{\Gamma}}(0) = 0,$$

which implies $u^* = 0$ at the equilibrium of the feedback system. Hence, (26) is proven by the fact that the equilibrium is asymptotically stable owing to the incremental passivity.

Next, we show (27). In the limit of $\tau \to 0$, coinciding with the singular perturbation approximation of κ, it approaches to the static controller given as

$$w = H\nabla\overline{G}(H^{\mathsf{T}} v).$$

This implies that the resultant feedback system is identical to $\hat{\Sigma}$, given as replacing F in (22) with $\hat{F}$ in (18). Note that $\overline{G}(-H^{\mathsf{T}} Cx)$ is convex. Thus, (27) follows from the monotonicity as shown in (20). ■

Theorem 2 shows that the additional implementation of the convex gradient controller G in (25) does not change the equilibrium (x^*, λ^*) of Σ from (22). Furthermore, the stability degree of the equilibrium can be improved in view of the stability index in Definition 3, as long as the time constant τ is sufficiently small. As for a suitable choice of τ, there can be a trade-off relation between the convergence rate to equilibria and the sensitivity to measurement noise. More specifically, a smaller value of τ would be desirable to improve the convergence rate whereas a larger value of τ would be desirable to suppress measurement noise in a larger frequency range.

IV. Numerical Examples

A. System Description

Let us consider giving the system dynamics as

$$\begin{cases} M\ddot{\zeta} + D\dot{\zeta} + K\zeta = Eu \\ y = E^{\mathsf{T}}\dot{\zeta} \end{cases}$$

where $M \succ 0$ and $D \succ 0$ denote mass and damper diagonal matrices, $K \succ 0$ denotes a spring stiffness matrix, and E is a matrix having a compatible dimension. This second-order system is often used as a primary model of rotor dynamics for power system stabilization [20], [21], called a swing equation, and it is incrementally passive since the input and output are collocated [9].

With the state of $x := [\zeta^{\mathsf{T}}, \dot{\zeta}^{\mathsf{T}}]^{\mathsf{T}}$, the system dynamics can be represented as Σ in (1) with

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M^{-1}E \end{bmatrix}, \quad C = \begin{bmatrix} 0 & E^{\mathsf{T}} \end{bmatrix}, \tag{28}$$

where $f(x) = Ax$. For this system, we consider the situation where output regulation by the integrators in (22) is performed in a decentralized fashion. To represent this, we give E in (28) as

$$E = \mathrm{diag}(\mathbf{1}_{n_i})_{i\in\mathbb{N}}, \quad \mathbb{N} := \{1,\dots,N\} \tag{29}$$

where $\mathbf{1}_n \in \mathbb{R}^n$ denotes the all-ones vector, N denotes the number of subsystems, and n_i denotes the dimension of the ith subsystem, satisfying $n = n_1 + \cdots + n_N$.

B. Convex Gradient Controller with Input Saturation

Next, we consider implementing a set of decentralized controllers that individually supervise clusters of several subsystems, i.e., a disjoint set of subsystems. To represent this, we define an aggregated output signal by

$$\hat{y}_l := \textstyle\sum_{i\in\mathcal{C}_l} y_i \tag{30}$$

where y_i denotes the ith element of y, and $\mathcal{C}_l \subset \mathbb{N}$ denotes a cluster defined as a consecutive index set such that

$$\textstyle\bigcup_{l\in\mathbb{L}} \mathcal{C}_l = \mathbb{N}, \quad \max \mathcal{C}_l < \min \mathcal{C}_{l+1}, \quad \mathbb{L} := \{1,\dots,L\}$$

for L denoting the number of clusters.

To demonstrate the output regulator design based on Theorem 2, we consider implementing a set of decentralized controllers subject to input saturation. To this end, we give the dynamics of a controller for the lth cluster as

$$\kappa_l : \begin{cases} \tau\dot{\xi}_l = -\nabla\mathit{\Gamma}_l(\xi_l) + k_l(\hat{y}_l^* + v_l) \\ w_l = k_l\xi_l, \end{cases} \tag{31}$$

where $k_l > 0$ denotes a controller gain, $\hat{y}_l^*$ denotes the aggregated reference signal defined similarly to (30), and

$$\nabla\mathit{\Gamma}_l(\xi_l) = \frac{\alpha_l}{2}\ln\frac{\xi_l^+}{\xi_l^-}, \quad \mathit{\Gamma}_l(\xi_l) := \frac{\alpha_l}{2}(\xi_l^+\ln\xi_l^+ + \xi_l^-\ln\xi_l^-)$$

for $\xi_l^+ := \alpha_l + \xi_l$ and $\xi_l^- := \alpha_l - \xi_l$. The set of controllers is connected with Σ by

$$v_l = -\hat{y}_l, \quad u_i = w_l, \quad i \in \mathcal{C}_l, \quad l \in \mathbb{L},$$

where $\hat{y}_l$ is defined as in (30) and u_i denotes the ith element of u.

As shown in Theorem 2, the asymptotic stability of (26) is guaranteed because $\mathit{\Gamma}_l$ is a strictly convex function satisfying $\nabla\mathit{\Gamma}_l(0) = 0$. Note that the controller state ξ_l is confined to the effective domain $\mathcal{X}_{G_l} = (-\alpha_l, \alpha_l)$. This implies that the control input is subject to the saturation of

$$|u_i| \le k_l\alpha_l, \quad \forall i \in \mathcal{C}_l. \tag{32}$$

In addition, we see that the linear approximation of κ_l in (31) becomes

$$\tau\dot{w}_l = -w_l + k_l^2(\hat{y}_l^* + v_l),$$

whose transfer function from $\hat{y}_l^* + v_l$ to w_l is $k_l^2/(1+\tau s)$. Thus, κ_l can be considered as a one-dimensional low-pass filter subject to input saturation. Furthermore, in the limit of $\tau \to 0$, κ_l approaches to the static controller

$$w_l = k_l\nabla\overline{\mathit{\Gamma}}_l\big(k_l(\hat{y}_l^* + v_l)\big) \tag{33}$$

where the conjugate function is given by

$$\nabla\overline{\mathit{\Gamma}}_l(\mu_l) = \alpha_l\tanh\frac{\mu_l}{\alpha_l}, \quad \overline{\mathit{\Gamma}}_l(\mu_l) := \alpha_l^2\ln\cosh\frac{\mu_l}{\alpha_l}.$$

From this expression, we see that the input saturation is represented as the hyperbolic tangent (sigmoid) function. As shown in Theorem 2, the static controller in (33) has the ability to improve the degree of stability in the sense of (27).

C. Simulation Results

We consider the network of 16 mass components, which yields a 32-dimensional system Σ in (1) with (28). The system network structure is depicted in Fig. 1, where the nodes represent the mass components and the edges represent the interconnection among them. The parameter matrices are given as $M = 1.25 \times I$, $D = 0.02 \times I$, and

$$K_{i,j} = \begin{cases} -0.3, & \text{if nodes } i \text{ and } j \text{ are connected,} \\ 0, & \text{otherwise,} \end{cases}$$
$$K_{i,i} = \begin{cases} 0.3 - \sum_{j=1,j\neq i}^{16} K_{i,j}, & i = 1, \\ -\sum_{j=1,j\neq i}^{16} K_{i,j}, & \text{otherwise,} \end{cases}$$

where $K_{i,j}$ denotes the (i,j)-element of K.

Let us regard the 32-dimensional system as the network of four subsystems, each of which is composed of four mass components, as shown in the dotted circles in Fig. 1. This corresponds to $N = 4$ and $n_i = 8$ in (29). In this setting, implementing the integrators as in (22), we plot the output signals y by the dashed-dotted lines in the upper subfigure of Fig. 2, where the reference signal is set to $y^* = \mathbf{1}_4$. From this figure, we see that the amplitude of oscillations is not made small in the time interval. This is due to the fact that the degree of inherent system stability is not very large.

To improve the convergence rate in the output regulation, we implement a set of dynamical controllers with the nonlinearity of input saturation, explained in Section IV-B. For the controllers κ_l in (31), we construct the clusters as $\mathcal{C}_1 = \{1,2\}$ and $\mathcal{C}_2 = \{3,4\}$, which lead to the aggregated output $\hat{y}_l$ as in (30). The controller parameters are given as $k_l = 0.8$ and $\alpha_l = 0.0625$ for $l = 1,2$, which result in the input saturation of $|u_i| \le 0.05$ as in (32). In this setting, the resultant outputs y and inputs u are plotted in the upper and lower subfigures of Fig. 2, respectively, where the thin solid lines correspond to the case of the static controller in (33) and the dashed lines correspond to the case of the dynamical one with the time constant of $\tau = 0.16$. From these figures, we see that the static and dynamical controllers result in the almost same output and input signals and both controllers make the convergence rate higher, in spite of the input saturation.

In addition, we show the results when giving the controller time constants as $\tau = 1$ and $\tau = 9$. The resultant outputs and inputs are plotted in Fig. 3 where the solid and dashed lines correspond to the cases of $\tau = 1$ and $\tau = 9$, respectively. From this figure, we see that the convergence rates become lower as increasing the value of time constants. As long as the measurement output is not contaminated with noise, we

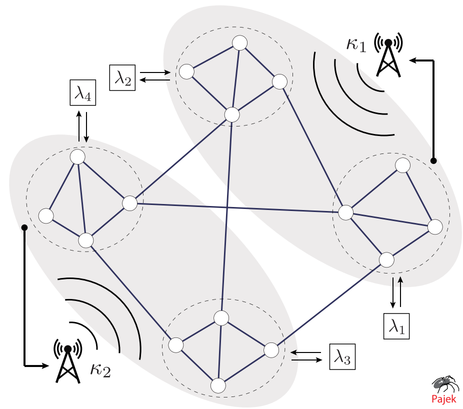

Fig. 1. System network structure and decentralized output regulators.

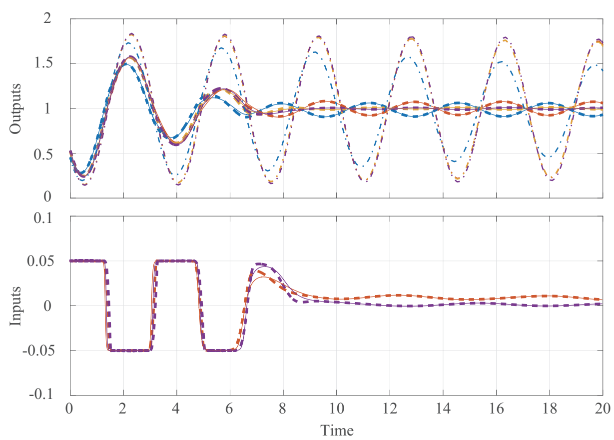

Fig. 2. Output and input signals in the cases of simple integration, static controller and dynamical controller with $\tau = 0.16$.

can improve the convergence rate, i.e., the degree of closed-loop stability, as giving a smaller time constant of controllers. Explicit consideration of measurement noise, which can be suppressed by the dynamical action (low-pass property) of controllers, is one of future works to pursue.

V. Concluding Remarks

In this paper, we have derived an equivalence condition for incremental passivity in terms of convex gradients. In particular, focusing on the class of incremental passive systems with a quadratic storage function, which can be transformed into a self-dual realization, we show that the convexity of potential functions is necessary and sufficient for the incremental passivity of systems whose vector field is given as the gradient of potential functions. Furthermore, we have analyzed the equilibrium of convex gradient systems via the convex conjugate defined by the Legendre-Fenchel transformation.

On the basis of these theoretical investigations, we have developed a design method of output regulators for incrementally passive systems. The effectiveness of the proposed method has been shown though a numerical example of decentralized output regulation in the frequency control of power systems. In this numerical example, we have demonstrated the improvement of convergence rates, while showing that a nonlinear low-pass filter subject to input saturation can be regarded as a convex gradient system.

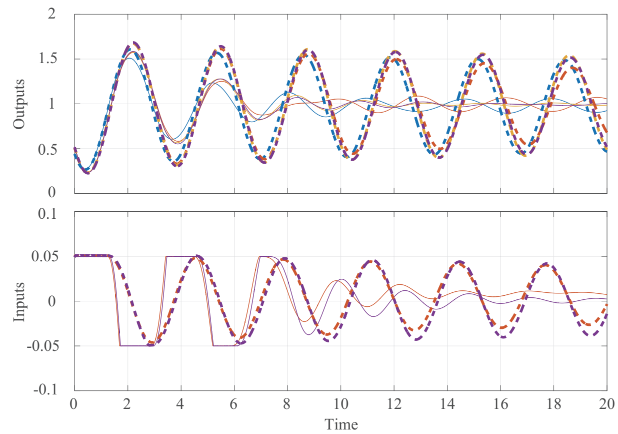

Fig. 3. Output and input signals in the cases of dynamical controllers with $\tau = 1$ and $\tau = 9$.

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