Projective State Observers for Large-Scale Linear Systems

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***Abstract*— In this paper, towards efficient state estimation for large-scale linear systems, we propose a novel framework of low-dimensional functional state observers, which we call a projective state observer, with the provision of a systematic design procedure. The projective state observer can be regarded as a generalization of functional state observers that is defined by means of the orthogonal projection taking into account the system controllability/observability in a quantitative manner. The efficiency of the proposed observer is shown through a numerical example for a reaction-diffusion system evolving over a directed complex network. Moreover, through this observer design, we discuss a trade-off relation between low-dimensionality and an observation error.**

I. Introduction

With the recent developments in engineering, the architecture of systems has been more complex and larger in scale. An appropriate state observer is needed for realizing state feedback control of large-scale systems. However, in the classical observer design methods, the resultant observer is necessary to have a dimension comparable with a system to be observed; see e.g., [1], [2], [3]. Thus, a high-dimensional observer is inevitably required for large-scale systems. In view of this, a design method of low-dimensional observers is crucial to deal with large-scale systems.

As related work, some methods for designing low-dimensional observers can be found in literature; see [4], [5]. In these methods, a state observer is often designed for a low-dimensional approximate model of systems that is obtained by a model reduction technique. For example, [5] proposes a design procedure of low-dimensional observers by using an approximant obtained by the balanced truncation [6]. However, in this method, the construction of approximate models is rather heuristic because the relation between the approximation error and observation error is not discussed theoretically.

To discuss the observation error for low-dimensional observers theoretically, we need to take into account some differences between the classical and low-dimensional observer design. For example, in the case of classical observers, such as the Luenberger-type observer, the external input, which is applied to both system and observer, does not have any influence on observation errors, i.e., the error dynamics is exactly uncontrollable. On the other hand, in the case of low-dimensional observers, the external input possibly excites observation errors, since the observer dynamics cannot exactly imitate the dynamical system behavior for the input signals.

Against this background, in the first half of this paper, we propose a novel framework of low-dimensional observers, which we call a *projective state observer*. The projective state observer can be regarded as a generalization of functional state observers that is defined by means of the orthogonal projection taking into account the system controllability/observability in a quantitative manner. Furthermore, we derive a tractable representation of the error system clarifying that not only an initial state estimation error but also the input signal and the initial value response of systems are relevant to the observation error.

In the second half of this paper, based on the error analysis in the first half, we propose a systematic design procedure for projective state observers. More specifically, we propose two types of design procedures for projective state observers that can estimate a set of specific states given in advance, and can estimate a kind of average behavior of systems, respectively. The former one is effective if we want to estimate a specific limited number of states. Although this type of observers is proposed in [7], the approach is different from the design procedure shown in this paper. This is because the approach is based on observer reduction where a reduced order observer approximates a full-state observer for the original system.

To explain the basic concept of the latter one, let us imagine the following situation: From a microscopic point of view, the behavior of fluid is realized by complex interaction among a huge number of molecules; on the other hand, from a macroscopic point of view, we can observe only a kind of average behavior of molecules. This fact suggests that the estimation of average behavior is essential to control the fluid behavior (i.e., large-scale systems) from a macroscopic point of view.

However, it should be noted that to find a set of average states that is suitable for the state estimation is nontrivial in general. To find such a set of average states, we apply the concept of clustered model reduction developed in [8], [9]. Consequently, we can design a projective state observer that can systematically find a set of states that is desirable for estimating the average system behavior. The efficiency of the two design procedures for projective state observers is shown through a numerical example for a reaction-diffusion system evolving over a directed complex network.

This paper is organized as follows: In Section II, we first give a mathematical formulation of projective state observers. Furthermore, deriving a tractable representation of the error system, we clarify differences between the design

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of classical functional state observers and that of projective state observers. In Section III-A, we describe a road map for systematic design of projective state observers. Then, in Section III-B and Section III-C, we propose two design procedures of projective state observers which can estimate a set of states and average behavior of the system, respectively. In Section IV, we show the efficiency of the proposed methods through a numerical example of a networked system. Finally, concluding remarks are provided in Section V.

Notation The following notation is used:

$\mathbb{R}$	set of real numbers
I_n	unit matrix of size $n \times n$
$M \prec \mathcal{O}_n$ ($M \succ \mathcal{O}_n$)	negative (positive) definiteness of a symmetric matrix $M \in \mathbb{R}^{n\times n}$
$M \preceq \mathcal{O}_n$ ($M \succeq \mathcal{O}_n$)	negative (positive) semidefiniteness of a symmetric matrix $M \in \mathbb{R}^{n\times n}$
$\mathrm{im}(M)$	range space spanned by the column vectors of a matrix M
$\mathrm{tr}(M)$	trace of a matrix M
$\|M\|_F$	the Frobenius norm of a matrix M

The $\mathcal{L}_2$-norm of a square integrable function $v(t) \in \mathbb{R}^n$ is defined by

$$\|v(t)\|_{\mathcal{L}_2} := \left(\int_0^\infty v^{\mathsf{T}}(t) v(t) dt \right)^{\frac{1}{2}}.$$

The $\mathcal{H}_\infty$-norm of a stable proper transfer matrix G and the $\mathcal{H}_2$-norm of a stable strictly proper transfer matrix G are respectively defined by

$$\|G(s)\|_{\mathcal{H}_\infty} := \sup_{\omega\in\mathbb{R}} \|G(j\omega)\|,$$

$$\|G(s)\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{tr}(G(j\omega)G^{\mathsf{T}}(-j\omega)) d\omega \right)^{\frac{1}{2}}$$

where $\|\cdot\|$ denotes the induced 2-norm.

II. Fundamentals of Projective State Observers

A. Preliminary Review of Functional State Observers

In this paper, we deal with the n-dimensional linear system described by

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}, \quad x(0) = x_0 \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m_u}$, $C \in \mathbb{R}^{m_y\times n}$, and $D \in \mathbb{R}^{m_y\times m_u}$. To simplify the arguments, we show results for stable systems. Similar results are available also for unstable systems. In Σ in (1), a measurement output signal is denoted by $y \in \mathbb{R}^{m_y}$. Furthermore, we define a signal to be estimated by

$$z = Sx \tag{2}$$

where $S \in \mathbb{R}^{m_z\times n}$.

In this notation, let us consider the n-dimensional (Luenberger-type) observer described by

$$O : \begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + H(y - \hat{y}) \\ \hat{y} = C\hat{x} + Du \end{cases}, \quad \hat{x}(0) = \hat{x}_0 \tag{3}$$

where the observer gain $H \in \mathbb{R}^{n\times m_y}$ is a design parameter. As being compatible with (2), we define the estimation of z by

$$\hat{z} = S\hat{x}. \tag{4}$$

In what follows, O is referred to as a functional state observer [2], [3] since z and $\hat{z}$ are defined as functions of x and $\hat{x}$.

By denoting the state error as $e := x - \hat{x}$, we obtain the error system with this functional state observer as

$$\mathcal{E} : \begin{cases} \dot{e} = (A - HC)e \\ \Delta = Se \end{cases}, \quad e(0) = e_0 \tag{5}$$

where

$$\Delta := z - \hat{z}, \quad e_0 := x_0 - \hat{x}_0$$

denote the observation error and the initial state error, respectively. Note that the observation error is a function of e_0, namely

$$\Delta = \Delta(t; e_0). \tag{6}$$

Usually, we design the observer gain H so that this error system has a desirable behavior. To regulate a convergence rate of observation errors, for a given constant $\delta \geq 0$, one can design H such that

$$\sup_{e_0\neq 0} \frac{\|\Delta(t; e_0)\|_{\mathcal{L}_2}}{\|e_0\|} \leq \delta. \tag{7}$$

In the next subsection, we define a lower-dimensional functional observer as a generalization of this n-dimensional functional state observer.

B. Error Analysis of Projective State Observers

In this subsection, for Σ in (1), we define an $\hat{n}$-dimensional functional state observer as a generalization of O in (3). More specifically, by means of the orthogonal projection [6], we define the $\hat{n}$-dimensional functional state observer by

$$O_P : \begin{cases} \dot{\hat{x}} = PAP^{\mathsf{T}}\hat{x} + PBu + H(y - \hat{y}) \\ \hat{y} = CP^{\mathsf{T}}\hat{x} + Du \end{cases}, \quad \hat{x}(0) = \hat{x}_0 \tag{8}$$

where the observer gain $H \in \mathbb{R}^{\hat{n}\times m_y}$ and the projection matrix $P \in \mathbb{R}^{\hat{n}\times n}$ satisfying $PP^{\mathsf{T}} = I_{\hat{n}}$ are design parameters. Without loss of generality, we assume $\hat{n} \leq n$. Similarly to (4), we define the estimation of z in (2) by

$$\hat{z} = SP^{\mathsf{T}}\hat{x}. \tag{9}$$

In the rest of this paper, we refer to this functional state observer O_P as a *projective state observer*.

To analyze the observation error for this projective state observer, we derive a tractable representation of the error system as follows:

Theorem 1: Let Σ in (1) be given with S in (2). Define O_P as in (8), and let $\Delta_P := z - \hat{z}$ with z and $\hat{z}$ defined as in (2) and (9). Then, it follows that

$$\mathcal{E}_P : \begin{cases} \dot{\xi} &= \mathcal{A}\xi + \mathcal{B}u \\ \Delta_P &= \mathcal{S}\xi \end{cases}, \quad \xi(0) = \begin{bmatrix} e_0 \\ x_0 \end{bmatrix} \tag{10}$$

where

$$e_0 := Px_0 - \hat{x}_0 \tag{11}$$

and

$$\mathcal{A} := \begin{bmatrix} PAP^{\mathsf{T}} - HCP^{\mathsf{T}} & (PA - HC)(I_n - P^{\mathsf{T}}P) \\ 0 & A \end{bmatrix}$$

$$\mathcal{B} := \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad \mathcal{S} := \begin{bmatrix} SP^{\mathsf{T}} & S(I_n - P^{\mathsf{T}}P) \end{bmatrix}.$$

Proof: By taking a state as $\hat{\mathcal{X}} := [\hat{x}^{\mathsf{T}}\ x^{\mathsf{T}}]^{\mathsf{T}}$, we have

$$\begin{cases} \dot{\hat{\mathcal{X}}} &= \hat{\mathcal{A}}\hat{\mathcal{X}} + \hat{\mathcal{B}}u \\ \Delta_P &= \hat{\mathcal{S}}\hat{\mathcal{X}} \end{cases}, \quad \hat{\mathcal{X}}(0) = \begin{bmatrix} \hat{x}_0 \\ x_0 \end{bmatrix}$$

where

$$\hat{\mathcal{A}} := \begin{bmatrix} PAP^{\mathsf{T}} - HCP^{\mathsf{T}} & HC \\ 0 & A \end{bmatrix}$$

$$\hat{\mathcal{B}} := \begin{bmatrix} PB \\ B \end{bmatrix}, \quad \hat{\mathcal{S}} := [-SP^{\mathsf{T}}\ S].$$

Define

$$T := \begin{bmatrix} -I_{\hat{n}} & P \\ & I_n \end{bmatrix} = T^{-1}.$$

From the similarity transformation of $T\hat{\mathcal{A}}T^{-1}, T\hat{\mathcal{B}}$ and $\hat{\mathcal{S}}T^{-1}$, the claim follows. ■

From Theorem 1, we notice that $\mathcal{E}_P$ in (10) corresponds to a generalized representation of the error system $\mathcal{E}$ in (5). This is because, if $P = I_n$, we have

$$\mathcal{A} = \begin{bmatrix} A - HC & 0 \\ 0 & A \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} S & 0 \end{bmatrix}$$

and

$$\xi(0) = \begin{bmatrix} e_0 \\ x_0 \end{bmatrix}, \quad e_0 = x_0 - \hat{x}_0.$$

Furthermore, it should be emphasized that Δ_P in (10) is a function of not only e_0 but also x_0 and u, namely

$$\Delta_P = \Delta_P(t; e_0, x_0, u), \tag{12}$$

which is clearly contrasted with Δ in (6) for the traditional functional state observer. An intuitive explanation on these error factors is as follows:

(i) the error due to the initial state error e_0 defined as in (11),

(ii) the error amplified by the initial value response of Σ, i.e., $y = Ce^{At}x_0$, and

(iii) the error amplified by the dynamical discrepancy of Σ and O_P for the external input u.

In conclusion, we see that the error factors (i), (ii) and (iii) should be taken into account for the design of projective state observers.

III. Design of Projective State Observers

A. A Road Map for Systematic Design

In what follows, we aim to find the design parameters P and H in (8) so that the observation error due to (i), (ii) and (iii) are suppressed as much as possible. Since the dynamics of the error system is linear, we can represent each error factor as

$$\Delta_P(t; e_0, 0, 0), \quad \Delta_P(t; 0, x_0, 0), \quad \Delta_P(t; 0, 0, u).$$

For the first one, similarly to (7), we consider regulating the convergence rate of the initial state error by introducing the index of

$$\sup_{e_0 \neq 0} \frac{\|\Delta_P(t; e_0, 0, 0)\|_{\mathcal{L}_2}}{\|e_0\|} \leq \delta \tag{13}$$

with a given constant $\delta \geq 0$.

Next, to see the effect of x_0 and u more explicitly, supposing that $e_0 = 0$, we derive the Laplace domain representation of the observation error due to the second and third factors as

$$\hat{\Delta}_P(s; x_0, u) := \Xi_{P,H}(s) X_P(s; x_0, u) \tag{14}$$

where

$$\Xi_{P,H}(s) := C_\Xi(sI_{\hat{n}} - A_\Xi)^{-1}B_\Xi + D_\Xi \tag{15}$$

with

$$A_\Xi := PAP^{\mathsf{T}} - HCP^{\mathsf{T}}, \quad B_\Xi := (PA - HC)\overline{P}^{\mathsf{T}}$$
$$C_\Xi := SP^{\mathsf{T}}, \quad D_\Xi := S\overline{P}^{\mathsf{T}}$$

and

$$X_P(s; x_0, u) := \overline{P}(sI_n - A)^{-1}[x_0 + Bu(s)] \tag{16}$$

with an orthogonal complement $\overline{P} \in \mathbb{R}^{(n-\hat{n})\times n}$ of $P \in \mathbb{R}^{\hat{n}\times n}$ such that

$$P^{\mathsf{T}}P + \overline{P}^{\mathsf{T}}\overline{P} = I_n. \tag{17}$$

From this expression, we can expect that the observation error due to the second and third factors will be small if the norms of $\Xi_{P,H}$ and X_P are small enough.

However, it should be noted that simultaneous design of P and H is difficult because $\Xi_{P,H}$ involves the design parameters in a bilinear fashion. To overcome this difficulty, we exploit the following facts:

- The parameter H appears in the system $\Xi_{P,H}$, but not in X_P.
- The system X_P involves the parameter $\overline{P}$ (or equivalently P), but not H.
- By a suitable choice of P, we can vanish D_Ξ in (15), which may directly increase the norm of $\Xi_{P,H}$.

Based on these facts, we first find P that minimizes the norm of X_P with the constraint of $D_\Xi = 0$, and then find H that minimizes the norm of $\Xi_{P,H}$. Taking this road map for the projective state observer design, we provide the following result:

Theorem 2: Let Σ in (1) be given with S in (2). For a constant $\alpha \geq 0$, define $\Phi \succeq \mathcal{O}_n$ such that

$$A\Phi + \Phi A^{\mathsf{T}} + BB^{\mathsf{T}} + \alpha I_n = 0. \tag{18}$$

Furthermore, take $P \in \mathbb{R}^{\hat{n}\times n}$ such that

$$\mathrm{im}\left(S^{\mathsf{T}}\right) \subseteq \mathrm{im}(P^{\mathsf{T}}), \quad PP^{\mathsf{T}} = I_{\hat{n}} \tag{19}$$

and

$$\mathrm{tr}(\Phi) - \mathrm{tr}(P\Phi P^{\mathsf{T}}) \leq \epsilon. \tag{20}$$

If there exist

$$\gamma > 0, \quad X \succ \mathcal{O}_{\hat{n}}, \quad Y \in \mathbb{R}^{\hat{n}\times m_y}$$

such that $X \prec \delta^2 I_{\hat{n}}$ and

$$\begin{bmatrix} \mathrm{sym}(XPAP^{\mathsf{T}} - YCP^{\mathsf{T}}) + PS^{\mathsf{T}}SP^{\mathsf{T}} & * \\ \overline{P}A^{\mathsf{T}}P^{\mathsf{T}}X - \overline{P}C^{\mathsf{T}}Y^{\mathsf{T}} & -\gamma I_{n-\hat{n}} \end{bmatrix} \prec \mathcal{O}_n \tag{21}$$

where $\mathrm{sym}(M) := M + M^{\mathsf{T}}$ and $\overline{P} \in \mathbb{R}^{(n-\hat{n})\times n}$ satisfying (17), then O_P in (8) with

$$H = X^{-1}Y \tag{22}$$

satisfies (13) for any $x_0 \in \mathbb{R}^n$ and $\hat{x}_0 \in \mathbb{R}^{\hat{n}}$, and

$$\|\Delta_P(t;0,0,u)\|_{\mathcal{L}_2}^2 + \alpha\|\Delta_P(t;0,x_0,0)\|_{\mathcal{L}_2}^2 \leq \gamma\epsilon \tag{23}$$

for the unit impulse input u and $x_0 \in \mathbb{R}^n$ such that $\|x_0\| = 1$, where e_0 and Δ_P are defined as in (11) and (12).

Proof: First, we evaluate $\|\Delta_P(t;0,0,u)\|_{\mathcal{L}_2}^2$ for the impulse input u. Based on the error system expression in (14), we have

$$\|\Delta_P(t;0,0,u)\|_{\mathcal{L}_2}^2 \leq \|\Xi_{P,H}(s)\|_{\mathcal{H}_\infty}^2 \|\overline{P}(sI - A)^{-1}B\|_{\mathcal{H}_2}^2$$

where $\Xi_{P,H}$ is defined as in (15). Substituting (22) into (21), we have

$$\begin{bmatrix} \mathrm{sym}(XPAP^{\mathsf{T}} - XHCP^{\mathsf{T}}) + PS^{\mathsf{T}}SP^{\mathsf{T}} & * \\ \overline{P}A^{\mathsf{T}}P^{\mathsf{T}}X - \overline{P}C^{\mathsf{T}}H^{\mathsf{T}}X & -\gamma I_{n-\hat{n}} \end{bmatrix} \prec \mathcal{O}_n. \tag{24}$$

Note that the first condition in (19) guarantees

$$D_\Xi = S\overline{P}^{\mathsf{T}} = 0.$$

Thus, from the bounded-real lemma [10], (24) guarantees $\|\Xi_{P,H}(s)\|_{\mathcal{H}_\infty} < \gamma$. Note that there exists $\Phi^{(1)} \succeq \mathcal{O}_n$ satisfying

$$A\Phi^{(1)} + \Phi^{(1)}A^{\mathsf{T}} + BB^{\mathsf{T}} = 0$$

for the stable system Σ. Using $\|\overline{P}(sI - A)^{-1}B\|_{\mathcal{H}_2}^2 = \mathrm{tr}(\overline{P}\Phi^{(1)}\overline{P}^{\mathsf{T}})$, we have

$$\|\Delta_P(t;0,0,u)\|_{\mathcal{L}_2}^2 \leq \gamma \mathrm{tr}(\overline{P}\Phi^{(1)}\overline{P}^{\mathsf{T}}).$$

Next, we evaluate $\|\Delta_P(t;0,x_0,0)\|_{\mathcal{L}_2}^2$. Note that there exists $\Phi^{(2)} \succeq \mathcal{O}_n$ satisfying $A\Phi^{(2)} + \Phi^{(2)}A^{\mathsf{T}} + I_n = 0$. Thus, using

$$x_0x_0^{\mathsf{T}} \preceq I_n \tag{25}$$

for any $x_0 \in \mathbb{R}^n$ such that $\|x_0\| = 1$, we have $\|\overline{P}(sI - A)^{-1}x_0\|_{\mathcal{H}_2}^2 \leq \mathrm{tr}(\overline{P}\Phi^{(2)}\overline{P}^{\mathsf{T}})$. Therefore

$$\begin{aligned}&\|\Delta_P(t;0,0,u)\|_{\mathcal{L}_2}^2 + \alpha\|\Delta_P(t;0,x_0,0)\|_{\mathcal{L}_2}^2 \\ &\quad\leq \gamma \mathrm{tr}\left(\overline{P}(\Phi^{(1)} + \alpha\Phi^{(2)})\overline{P}^{\mathsf{T}}\right).\end{aligned}$$

From the Lyapunov theorem [11], Φ given in (18) satisfies $\Phi = \Phi^{(1)} + \alpha\Phi^{(2)}$. In addition, the condition in (20) yields

$$\mathrm{tr}\left(\overline{P}(\Phi^{(1)} + \alpha\Phi^{(2)})\overline{P}^{\mathsf{T}}\right) \leq \epsilon.$$

Thus, (23) follows. Finally, we show (13). To describe the time evolution of $\eta(t) := \Delta_P(t;e_0,0,0)$, we consider the system given by

$$\begin{cases} \dot{e} = (PAP^{\mathsf{T}} - HCP^{\mathsf{T}})e \\ \eta = SP^{\mathsf{T}}e \end{cases}, \ e(0) = e_0.$$

From $(1,1)$ block of (21), this system admits a Lyapnov function $V(e) := e^{\mathsf{T}}Xe$ such that $\dot{V}(e(t)) < -\eta^{\mathsf{T}}(t)\eta(t)$. Integrating this inequality over $[0,\infty)$ and utilizing $V(e(\infty)) = 0$, we have

$$\|\eta(t)\|_{\mathcal{L}_2}^2 < V(e(0)) \leq \|e_0\|^2\|X\|_2$$

for any $e_0 \in \mathbb{R}^{\hat{n}}$. Hence, (13) follows from $X \prec \delta^2 I_{\hat{n}}$. ■

Theorem 2 provides an explicit error bound for projective state observers. As shown in the proof of this theorem, we measure the effect of u in terms of the $\mathcal{H}_2$-norm, with similar results available for the case of the $\mathcal{H}_\infty$-norm. One possible approach to find P in the $\mathcal{H}_\infty$-norm evaluation is *system tridiagonalization*; see [8] for details.

B. Design of Projective State Observers for Specific State Estimation

In this subsection, we propose a procedure to construct a projective state observer with S given in advance. This projective state observer estimates a subset of the state-space of Σ, which is specified by z in (2). Such a projective state observer is efficient when we want to estimate a limited number of states, as in the use of traditional functional state observers.

In this situation, we can find $P \in \mathbb{R}^{\hat{n}\times n}$ such that (19) and (20) in the following manner: First, we find the set $\{(\lambda_i, v_i)\}_{i\in\{1,\ldots,n\}}$ of all eigenpairs of Φ, supposing that $\lambda_i \geq \lambda_{i+1}$ and $\|v_i\| = 1$ without loss of generality. Next, we find minimum $k \in \{1,\ldots,n\}$ such that

$$\lambda_{k+1} + \cdots + \lambda_n \leq \epsilon \tag{26}$$

and construct $V = [v_1,\ldots,v_k] \in \mathbb{R}^{n\times k}$. Finally, by the Gram-Schmidt process, we derive P such that

$$\mathrm{im}(P^{\mathsf{T}}) = \mathrm{im}([V, S^{\mathsf{T}}]). \tag{27}$$

Based on this, supposing that Σ in (1) with S in (2) is given, we summarize the design procedure of an $\hat{n}$-dimensional projective state observer O_P such that (13) and

$$\|\Delta_P(t;0,0,u)\|_{\mathcal{L}_2}^2 + \alpha\|\Delta_P(t;0,x_0,0)\|_{\mathcal{L}_2}^2 \leq \rho \tag{28}$$

where $\rho \geq 0$ is a design parameter.

1) Fix an initial value of $\epsilon \geq 0$.
2) Find $P \in \mathbb{R}^{\hat{n}\times n}$ such that (19) and (20) by the procedure above.
3) For a given δ, solve (21) while minimizing γ where $\overline{P}$ is constructed by Gram-Schmitt process.

4) If (21) is infeasible or $\rho < \gamma\epsilon$, then take larger δ and ρ and back to 2).
5) Compute H by (22) and construct a low-dimensional functional observer O_P in (8).

It should be noted that since the number of decision variables of LMI given by (21) is $\frac{1}{2}\hat{n}(\hat{n}-1)+\hat{n}m_y$, this design procedure is computationally tractable if $\hat{n}$ is small.

C. Design of Projective State Observers for Average State Estimation

In this subsection, we construct a projective state observer with S in (2) given by

$$S = P. \tag{29}$$

This implies that S is also a design parameter of projective state observers. In this setting, the condition in (19) is automatically satisfied. Furthermore, z and $\hat{z}$ in (2) and (9) are clearly given by $z = Px$ and $\hat{z} = \hat{x}$. It should be emphasized that, unless we impose a meaningful structure on P, the estimated signal $\hat{z}$ has no physical meaning. To accomplish the state estimation in a suitable sense, we impose the following block-diagonal structure on P:

Definition 1: The family of an index set $\{\mathcal{I}_{[l]}\}_{l\in\mathbb{L}}$ for $\mathbb{L} := \{1,\ldots,L\}$ is called a *cluster set*, whose element is referred to as a cluster, if each element $\mathcal{I}_{[l]}$ is a disjoint subset of $\{1,\ldots,n\}$ and it satisfies

$$\bigcup_{l\in\mathbb{L}} \mathcal{I}_{[l]} = \{1,\ldots,n\}.$$

Then, an *aggregation matrix* compatible with $\{\mathcal{I}_{[l]}\}_{l\in\mathbb{L}}$ is defined by

$$P := \mathrm{diag}(p_{[1]},\ldots,p_{[L]})\Pi \in \mathbb{R}^{L\times n} \tag{30}$$

with the permutation matrix

$$\Pi := [e^n_{\mathcal{I}_{[1]}},\ldots,e^n_{\mathcal{I}_{[L]}}]^{\mathsf{T}} \in \mathbb{R}^{n\times n}, \quad e^n_{\mathcal{I}_{[l]}} \in \mathbb{R}^{n\times|\mathcal{I}_{[l]}|} \tag{31}$$

and $p_{[l]} \in \mathbb{R}^{1\times|\mathcal{I}_{[l]}|}$ such that $\|p_{[l]}\| = 1$.

Based on this notion of clustering, the authors have solved a structured model reduction problem for networked systems in [8], [9]. Note that, if P has the specific structure shown in (30), then the estimated signal $\hat{z}$ can be interpreted as a *weighted average* of states of Σ. In particular, if $p_{[l]}$ is in the form of

$$p_{[l]} = \frac{[1,\ldots,1]}{\|[1,\ldots,1]\|} \in \mathbb{R}^{1\times|\mathcal{I}_{[l]}|}, \tag{32}$$

then $\hat{z}$ corresponds to an average state in the sense of

$$z_l = \frac{1}{\sqrt{|\mathcal{I}_{[l]}|}}\sum_{\forall i\in\mathcal{I}_{[l]}} x_i, \quad l\in\mathbb{L} \tag{33}$$

where z_l (resp. x_i) is the lth element of z (resp. the ith element of x). A method to achieve usual averaging is described in Remark 1. In what follows, for simplicity, we only consider the case of (32).

By the design procedure same as in Section III-B, let us construct a projective state observer that estimates the average behavior of systems. To this end, we can use the following fact:

Lemma 1: Let Σ in (1) be given, and define $\Phi \succeq \mathcal{O}_n$ such that (18) for a constant $\alpha \geq 0$. If, for each $l\in\mathbb{L}$, there exists a row vector $\phi_{[l]} \in \mathbb{R}^{1\times n}$ such that

$$\left\|(e^n_{\mathcal{I}_{[l]}})^{\mathsf{T}}\Phi_{\frac{1}{2}} - p^{\mathsf{T}}_{[l]}\phi_{[l]}\right\|_F \leq |\mathcal{I}_{[l]}|^{\frac{1}{2}}\theta \tag{34}$$

where $p_{[l]} \in \mathbb{R}^{1\times|\mathcal{I}_{[l]}|}$ satisfying $\|p_{[l]}\| = 1$ and $\Phi_{\frac{1}{2}}$ denotes a Cholesky factor such that $\Phi = \Phi_{\frac{1}{2}}\Phi^{\mathsf{T}}_{\frac{1}{2}}$, then it follows that

$$\mathrm{tr}(\Phi) - \mathrm{tr}(P\Phi P^{\mathsf{T}}) \leq \theta^2\left(\textstyle\sum_{l=1}^{L}|\mathcal{I}_{[l]}|(|\mathcal{I}_{[l]}|-1)\right)$$

where P is defined as in (30).

Proof: See [9]. ■

This lemma shows that, if we find a cluster set $\{\mathcal{I}_{[l]}\}_{l\in\mathbb{L}}$ such that (34), then the value of ϵ in Theorem 2 can be taken as

$$\epsilon = \theta^2\left(\textstyle\sum_{l=1}^{L}|\mathcal{I}_{[l]}|(|\mathcal{I}_{[l]}|-1)\right).$$

Remark 1: The estimated signal z given by S and P in (29) and (30) with (32) implies an average state in the sense of (33). Alternatively, if we take

$$S = DP, \quad D := \mathrm{diag}\left(\frac{1}{\sqrt{|\mathcal{I}_1|}},\ldots,\frac{1}{\sqrt{|\mathcal{I}_L|}}\right) \in \mathbb{R}^{L\times L} \tag{35}$$

then z implies an average state in the sense of

$$z_l = \frac{1}{|\mathcal{I}_{[l]}|}\sum_{\forall i\in\mathcal{I}_{[l]}} x_i, \quad l\in\mathbb{L}. \tag{36}$$

IV. Numerical Simulation

In this section, we show the efficiency of the proposed projective state observer through a numerical example. We deal with a 1000-dimensional reaction-diffusion system evolving over a complex network model, called the Dorogovtsev model [12].

First, we show the design result of a projected state observer for the specific state estimation proposed in Section III-B. Let the input signal $u \in \mathbb{R}^5$ be applied to the states of some five nodes, and the measurement output signal $y \in \mathbb{R}^{10}$ is obtained as the states of some 10 nodes. Moreover, the evaluation output $z \in \mathbb{R}^{20}$ is given as the states of some 20 nodes, which are different from the measurement output nodes. Based on the procedure shown in Section III-B, P and H in (8) are constructed for several values of ρ while taking $\alpha = 0$ and $\delta = 2.0$. Note that we are required to take at least $\hat{n} \geq 20$ to satisfy (19) in this setting.

In Fig. 1, we plot the resultant dimension of projective state observers versus the value of ρ in (28). Furthermore, Fig. 2 shows the resultant performance of the input response $\Delta_P(t;0,0,u)$ versus the value of ρ in (28). In addition, the resultant performance of the initial state error are about 2.0 for all ρ. These results show that there is a trade-off

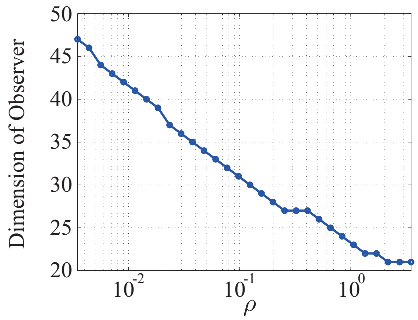

Fig. 1. Dimension of observer versus ρ.

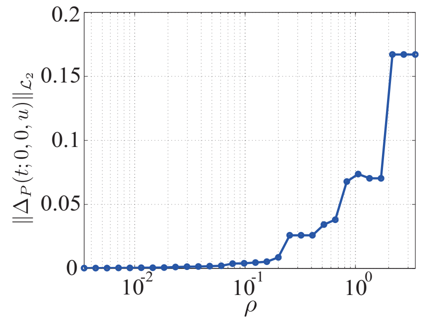

Fig. 2. Performance of input response versus ρ.

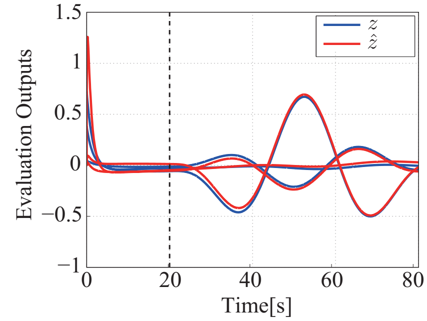

Fig. 3. Trajectories of z and $\hat{z}$.

relation between the estimation performance and the low-dimensionality of observers. Namely, the estimation performance is improved in compensation for increasing the dimension of observers.

Next, we verify the estimation performance of projective state observers by comparing $z(t)$ with $\hat{z}(t)$. In Fig. 3, we plot only three elements of z and $\hat{z}$ since the other elements behave similarly. To obtain this figure, a random input signal is applied over $t \in [20, 80]$ and x_0 is given randomly and $\hat{\xi}_0 = 0$. The resultant dimension of the projective state observer is $\hat{n} = 47$. From this figure, we can confirm that the observation error relevant to both input and initial state response are small.

Finally, we show the design result of projective state observers for average state estimation proposed in Section III-C. The observer is constructed by the procedure shown in Section III-B while taking $\alpha = 0$, $\delta = 5.0$ and $\rho = 4.3$. In addition, S is constructed by (35), which implies that z estimates an average state in the sense of (36). The resultant dimension of the obtained observer is $\hat{n} = 5$, which implies that the resultant number of clusters is $L = 5$. In Fig. 4, we plot all trajectories of x and $\hat{z}$ where a random input signal is applied over $t \in [0, 140]$. The legends are as follows: the trajectory of $\hat{z}$ is depicted by the dotted lines with the circles where $\hat{z}_l$ is color-coded according to $l \in \{1, \ldots, 5\}$. In addition, x_i for each $i \in \mathcal{I}_{[l]}$ is also color-coded according to its cluster index $l \in \{1, \ldots, 5\}$. From this figure, we can confirm that each trajectory of five elements of $\hat{z}$ is around the center of colored trajectory sets of x. Moreover, the resultant estimation error is $\|z - \hat{z}\|^2_{\mathcal{L}_2} = 1.3 \times 10^{-2}$, which implies that $\hat{z}$ estimates an average state in the sense of (36) efficiently. As demonstrated in this numerical example, the proposed projective state observer can efficiently find and estimate the average behavior of networked systems.

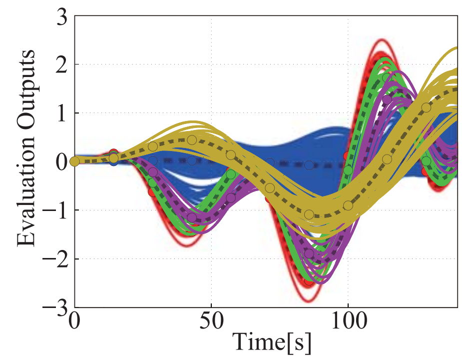

Fig. 4. Trajectories of $\hat{z}$ and x.

V. Conclusion

In this paper, we proposed a novel framework of low-dimensional functional state observers, called projective state observers. In addition, deriving a tractable representation of the error system, we clarified differences between the design of classical functional state observers and that of projective state observers. Based on the error analysis, we provided a systematic design procedure. For two types of projective state observers: the first can estimate a set of states of the system and the second can find and estimate the average behavior of systems. The efficiency of the proposed design procedure is shown through a numerical simulation.

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